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Higher spin symmetry and $\mathcal{N} = 4$ SYM

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Abstract: We assemble the spectrum of single-trace operators in free $\mathcal{N} = 4$ SU(N) SYM theory into irreducible representations of the Higher Spin symmetry algebra $\mathfrak{hs}(2,2|4)$. Higher Spin representations or *YT-pletons* are associated to Young tableaux (YT) corresponding to representations of the symmetric group compatible with the cyclicity of color traces. After turning on interactions $g_{\mathrm{YM}} \neq 0$, YT-pletons decompose into infinite towers of representations of the superconformal algebra $\mathfrak{psu}(2,2|4)$ and anomalous dimensions are generated. We work out the decompositions of tripletons with respect to the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2,2|4)$ and compute their anomalous dimensions to lowest non-trivial order in $g^2_{\mathrm{YM}}N$ at large N. We then focus on operators/states sitting in semishort multiplets of $\mathfrak{psu}(2,2|4)$. By passing them through a semishort-sieve that removes superdescendants, we derive compact expressions for the partition function of semishort primaries.

Keywords: AdS-CFT and dS-CFT Correspondence, Space-Time Symmetries, Supersymmetry and Duality.

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Contents

1. Introduction

Pushing Maldacena's conjecture [1] to its extreme consequences, one is led to conclude that free $\mathcal{N}=4$ super Yang-Mills theory (SYM) with SU(N) gauge group should be holographically dual to type IIB superstring on an extremely curved $AdS_5\times S^5$ [2]–[5]. The Hagedorn growth of single-trace gauge-invariant SYM operators at large N precisely reproduces (tree level) string expectations and one is led to take the limit seriously [2]–[8] and try to match the spectra on the two sides of the correspondence. At vanishing gauge coupling constant ($g_{\rm YM}=0$) $\mathcal{N}=4$ SYM theory develops a higher spin (HS) symmetry $\mathfrak{hs}(2,2|4)$. One thus expects the same should happen in the tensionless limit [9]–[14] or rather at some very small radius of order $R\approx\sqrt{\alpha'}$ [15]–[20] for the type IIB superstring on

$AdS_5 \times S^5$ (see [21]–[25] for studies of higher spin gauge theories in various dimensions). Despite some progress [26]–[28], string quantization in the presence of RR backgrounds is poorly understood in general, let alone at large curvatures, and one has to devise some alternative strategy for the time being.

In [29] a precision test of the correspondence was carried out by first extrapolating the naive Kaluza-Klein (KK) reduction on S^5 of the type IIB superstring spectrum from ten dimensions to the point of enhanced HS symmetry and then postulating a mass formula for the resulting string excitations that could account for the appearance of the expected massless HS gauge fields. The impressive agreement with single-trace SYM operators at large N up to dimension 4, including those belonging to genuinely long supermultiplets, led us to suspect that one could do better and find a more accurate energy formula valid for all superstring states at the point of HS symmetry enhancement. Indeed by relying on the BMN limit [30] and extrapolating the plane-wave frequencies down to finite J (at $g_{\rm YM} = 0$!) such a formula was found [31]

$$\Delta = J + \nu \,, \tag{1.1}$$

where $\nu = \sum_n N_n$ is the string occupation number and J the charge under an $\mathfrak{so}(2)$ subgroup of a hidden $\mathfrak{so}(10)$ symmetry [31, 32] that organizes the KK string spectrum.[1] Despite its simplicity, (1.1) encompasses the correct 'energies' for all string states to match those of SYM operators up to dimension $\Delta = 10$ together with their superdescendants that neatly assemble into (long) supermultiplets of $\mathfrak{psu}(2,2|4)$.[2]

The aim of this paper is to rearrange the SYM / string spectrum into multiplets of the higher spin (HS) extension $\mathfrak{hs}(2,2|4)$ of the superconformal group. Representations of $\mathfrak{hs}(2,2|4)$ can be built out of multiple tensor products of singleton multiplets. The singleton of $\mathfrak{hs}(2,2|4)$ turns out to coincide with the singleton of $\mathfrak{psu}(2,2|4)$ with vanishing central charge, that consists of the fundamental SYM fields together with their derivatives [25, 15, 34, 35]. In the absence of abelian factors in the gauge group, the singleton does not give rise to well defined scaling operators. In the holographic description it corresponds to the low-lying open-string excitations that cannot propagate in the bulk of AdS. Its (gauge invariant) composites which correspond to closed-string excitations can [36]–[38]. The symmetric product of two singletons gives rise to the HS 'massless' doubleton containing all twist 2 gauge-invariant operators and their superpartners. They are dual to the HS gauge fields and their superpartners in the bulk. More precisely, the symmetric doubleton product decomposes into an infinite number of $\mathfrak{su}(2,2|4)$ multiplets [34, 35, 15] and [39]–[42]

$$(\mathcal{V}_F \times \mathcal{V}_F)_S = \sum_{n=0}^{\infty} \mathcal{V}_{2n} \,. \tag{1.2}$$

[1] While this paper was being published an interesting paper by Itzhak Bars appeared on the archive [33] that discussed how the spectrum of higher spin currents in $N = 4$ SYM could be related to a particular gauge fixing of the two-time superstring and reviewed previous work [32] where the higher dimensional origin of the SO(10) symmetry had been advocated.

[2] The upper bound of $\Delta = 10$ is imposed on us by computer capabilities.

Here $\mathcal{V}_F$ denotes the singleton and $\mathcal{V}_{2n\geq 2}$ are semishort current multiplets with primaries transforming as singlets of SU(4) and carrying spin $2n-2$. For instance $\mathcal{V}_2$ denotes the (semishort) $\mathcal{N}=4$ Konishi multiplet [43, 44]. Finally, $\mathcal{V}_0$ is the $\frac{1}{2}$-BPS supercurrent multiplet. The anti-symmetric doubletons with odd spin $\mathcal{V}_{2n+1}$ do not appear in the free SYM spectrum due to the cyclicity of the trace but play a role in the interactions.

As we will see, the pattern persists for higher tensor multiplets. For gauge group SU(N), the tensor product of L singletons decomposes into representations of $\mathfrak{hs}(2,2|4)$, that may be termed *YT-pletons* since they are completely classified by those Young tableaux (YT) with L boxes that are compatible with the cyclicity of the trace. At large N, mixing among single and multi trace operators is suppressed [45]–[49]. It is impossible anyway at $L=3$, where we will find only two 'massive' representations: the totally symmetric one including the first KK recurrence of the HS 'massless' doubleton, and the totally antisymmetric one, always present, that includes part of the lower spin Goldstone / Stückelberg fields needed for the Higgsing of the HS 'massless' gauge fields when departing from the HS symmetry enhancement point [34, 35, 15, 29]. For other gauge groups, such as SO(N) or Sp($2N$), one has to also take into account the symmetry under transposition that is holographically dual to a combination of worldsheet parity, spacetime inversion and fermion parity [50]–[53]. At $L=3$ this projects out the completely symmetric YT tableau leaving only the completely antisymmetric "Goldstone" multiplet. Related aspects of the SYM spectrum have been studied in [54]–[56] and [8].

In the boundary theory, turning on interactions ($g_{\text{YM}}\neq 0$) breaks the HS symmetry down to the superconformal supergroup $\mathfrak{psu}(2,2|4)$. As a result, both massless and massive representations of $\mathfrak{hs}(2,2|4)$ typically decompose into infinite series of $\mathfrak{psu}(2,2|4)$ supermultiplets. Massive representations of HS symmetry algebras have not been much studied in the past [57, 58]. Here we present the simplest occurrences of massive representations of $\mathfrak{hs}(2,2|4)$. They will play a crucial role in the Pantagruelic Higgs mechanism ('Grande Bouffe') in the AdS bulk that gives masses to all HS gauge fields except the graviton and its superpartners.

In a (superconformal) quantum field theory the violation of a symmetry due to quantum effects reflects into the corresponding current acquiring an anomalous dimension [59]–[64]. Anomalous dimensions can be computed either by *old-fashion* QFT methods [65]–[71] and [40, 41, 48, 68] or by *brand-new* techniques based on the identification of the planar dilatation operator [72] with the hamiltonian of an integrable super-spin chain [73]–[76]. Non planar dynamics is described by a spin chain with non-local interactions accounting for joining and splitting of SYM traces [72, 77]. Here we apply the spin chain techniques to determine the one-loop anomalous dimensions for some operators consisting of three constituent fields in $\mathcal{N}=4$ SYM.

The plan of the paper is as follows. In section 2 we discuss the algebra $\mathfrak{hs}(2,2|4)$, the HS extension of the superconformal algebra $\mathfrak{psu}(2,2|4)$, and its representations. We consider first the case of $\mathfrak{hs}(1,1)$, the HS extension of $\mathfrak{su}(1,1)\sim\mathfrak{sl}(2)$, spanned in $\mathcal{N}=4$ SYM by a single (complex) scalar field and its derivatives in a given (complex) direction. This truncation illustrates already the main features of the HS representation theory which apply

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to HS extensions of Poincaré and superconformal algebras in any dimension.[3] In particular we argue that irreducible representations of $\mathfrak{hs}(2,2|4)$ are in one-to-one correspondence with Young tableaux built out of singletons. Only a subset of these representations survives tracing over color indices.

In section 3 we describe how the spectrum of operators in free $\mathcal{N}=4$ SYM can be assembled into irreducible representation of $\mathfrak{hs}(2,2|4)$. We determine via Polya theory the set of Young tableaux surviving the tracing over $\mathrm{SU}(N)$ gauge indices and display the $\mathfrak{psu}(2,2|4)$ content of the first occurrences of massive HS representations at 'twist' 3. Higher L-pletons involve more complicated decompositions spanning several infinite towers of $\mathcal{N}=4$ multiplets.

In section 4 we restrict to states sitting in semishort multiplets of the $\mathcal{N}=4$ SCA. Disposing of superdescendants by means of a semishort-sieve, we derive compact expressions for $\mathcal{Z}^{\rm short}_{\rm suprim}$, the partition function of BPS and semishort primaries. In section 5 we turn on interactions, i.e. a small non-vanishing SYM coupling $g_{\rm YM}\neq 0$, that break $\mathfrak{hs}(2,2|4)$ down to $\mathfrak{psu}(2,2|4)$ and compute the anomalous dimensions of tripletons to lowest non-trivial order in $g^2_{\rm YM}N$ at large N. Finally, in section 6, we conclude with some comments on L-pletons and integrability. Appendix A introduces a unifying notation for $\mathcal{N}=4$ UIR's and shortenings originally discussed in [82]. Appendices B and C collect other useful formulae.

2. The higher spin algebra and its representations

At vanishing gauge coupling constant ($g_{\rm YM}=0$) the SCA $\mathfrak{psu}(2,2|4)$ of $\mathcal{N}=4$ SYM theory gets enhanced to the HS symmetry algebra $\mathfrak{hs}(2,2|4)$ [34, 35, 15, 23, 83, 84] and [62]–[64]. The HS symmetry algebra is generated by an infinite set of conserved currents of arbitrarily high (even) spin $s=2n$ associated to totally symmetric and traceless tensors

$$\mathcal{J}_{\mu_1\ldots\mu_{2n}} = \mathrm{Tr}\,\varphi^i\partial_{(\mu_1}\ldots\partial_{\mu_{2n})}\varphi_i+\cdots\,, \tag{2.1}$$

and their superpartners. Together with the lowest ultra-short $\frac{1}{2}$-BPS multiplet that contains the unbroken currents of the superconformal algebra $\mathfrak{psu}(2,2|4)$, the infinite tower of HS multiplets builds a single *massless* multiplet of the HS algebra $\mathfrak{hs}(2,2|4)$, the *doubleton* (1.2). The doubleton collects all gauge-invariant operators built from two SYM elementary fields $\{A_\mu,\lambda^\alpha_A,\bar\lambda^A_{\dot\alpha},\varphi^i\}$ and derivatives thereof modulo field equations. In general, all states belonging to a HS multiplet have a common length L, i.e. number of constituents or 'partons' ($L=2$ for the doubleton), since any linearly realized symmetry at $g_{\rm YM}=0$ preserves the number of letters.[4] However, for a given length $L>2$, several HS multiplets appear. In [34], the $\mathfrak{psu}(2,2|4)$ content of the HS massless multiplet was determined, and the HS gauge theory realizing the algebra $\mathfrak{hs}(2,2|4)$ on AdS_5 was formulated at the linearized level.

[3]HS algebras in other dimensions are supported by non-conformal free SYM theories living on Dp-branes are their gravity duals on warped AdS geometries [78]–[81].

[4]A closely related notion is the 'twist' $\tau=\Delta-s$, with Δ the scaling dimension and s the spin. $L=\tau$ for semishort primaries.

Here we consider *massive* representations[5] of $\mathfrak{hs}(2,2|4)$. As we will see they play a crucial role in the decomposition of the free $\mathcal{N}=4$ SYM spectrum on the boundary and in the Pantagruelic Higgs mechanism in the AdS bulk. Although our discussion will focus on $\mathfrak{hs}(2,2|4)$ for its relevance to $\mathcal{N}=4$, the analysis can be adapted to HS extensions of superconformal or Poincaré groups in other dimensions. To this purpose, it is instructive to start describing representations of the simplest HS algebra $\mathfrak{hs}(1,1)$, the HS extension of $\mathfrak{su}(1,1)\sim\mathfrak{sl}(2)$ [25]. This subalgebra, generated by a single derivative, is part of any HS algebra independently of the dimension. As we will see later on, the discussion of $\mathfrak{hs}(2,2|4)$, like any other HS extension, is an almost straightforward generalization of this simple case.

The algebra $\mathfrak{hs}(2,2|4)$ and its irreducible representations, or *YT-pletons*, are subsequently discussed in subsection 2.2. The decomposition of YT-pletons into an infinite sum of irreducible representations of the $\mathcal{N}=4$ superconformal subalgebra $\mathfrak{psu}(2,2|4)$ will be further discussed in section 3.

2.1 Representations of $\mathfrak{hs}(1,1)$

Here we describe the representations of the HS algebra $\mathfrak{hs}(1,1)$. This algebra is realized on a single complex scalar (in the adjoint of the gauge group) and its derivatives along a chosen (complex) direction. The aim of this section is to establish a one-to-one correspondence between irreducible representations of $\mathfrak{hs}(1,1)$ and Young tableaux made out of singletons. This section can be read independently of the rest of the paper and applies to HS algebras in any dimension containing $\mathfrak{hs}(1,1)$ as a subgroup.

$\mathfrak{sl}(2)$. We start by considering the $\mathfrak{sl}(2)$ subalgebra:

$$[J_-,J_+]=2\,J_3\,,\qquad [J_3,J_\pm]=\pm J_\pm\,. \tag{2.2}$$

This algebra may be represented in terms of oscillators

$$J_+=a^\dagger+a^\dagger a^\dagger a\,,\qquad J_3=\frac{1}{2}+a^\dagger a\,,\qquad J_-=a\,, \tag{2.3}$$

where, as usual, $[a,a^\dagger]=1$. For our purpose it is convenient to work in the space of functions $f(a^\dagger)$, wherein $a=\partial/\partial a^\dagger$. $\mathfrak{sl}(2)$ highest weight states (HWS's) are defined by

$$J_-f(a^\dagger)|0\rangle=0\quad\Rightarrow\quad f(a^\dagger)=1\,. \tag{2.4}$$

Any state $(a^\dagger)^n|0\rangle$ in this defining representation can be generated from its HWS $|0\rangle$ by acting with J_+^n. Therefore $f(a^\dagger)$ defines a single irreducible representation of $\mathfrak{sl}(2)$. For later purposes, we call this representation *singleton* and denote it by V_F. The $\mathfrak{sl}(2)$ spin of this representation is $-J_3\,|0\rangle=-\frac{1}{2}|0\rangle$. In $\mathcal{N}=4$ SYM, the components of the $\mathfrak{sl}(2)$ singleton may be chosen in $\mathfrak{psu}(2,2|4)/\mathfrak{sl}(2)$ different ways. In particular, the HWS can

[5] At $g_{\rm YM}=0$, the lowest cases ($L=3,4$) still contain some marginal scalar operators dual to some massless scalars in bulk.

be identified with the (complex) scalar $Z = \varphi^5 + i\varphi^6$ and its $\mathfrak{sl}(2)$ descendants can be generated by the action of the derivative along a chosen complex direction $\mathcal{D} = D_1 + iD_2$,

$$(a^\dagger)^n|0\rangle \quad \leftrightarrow \quad \mathcal{D}^n Z\,. \tag{2.5}$$

In a similar way, the tensor product of L singletons may be represented in the space of functions $f(a^\dagger_{(1)}, a^\dagger_{(2)}, \dots, a^\dagger_{(L)})$ with $a^\dagger_{(s)}$ acting on the $s^{\rm th}$ site. The resulting representation is no longer irreducible. This can be seen by looking for $\mathfrak{sl}(2)$ HWS's

$$J_-\, f(a^\dagger_{(1)}, \dots, a^\dagger_{(L)}) = \sum_{s=1}^{L} \partial_s\, f(a^\dagger_{(1)}, \dots, a^\dagger_{(L)}) = 0\,. \tag{2.6}$$

with $\partial_s = \frac{\partial}{\partial a^\dagger_{(s)}}$. There are indeed several solutions to these equations given by all possible functions of the differences $f_L(a^\dagger_{(s)} - a^\dagger_{(s')})$. A basis for $\mathfrak{sl}(2)$ HWS's can be chosen as

$$|j_1, \dots, j_{L-1}\rangle = (a^\dagger_{(L)} - a^\dagger_{(1)})^{j_1} (a^\dagger_{(L)} - a^\dagger_{(2)})^{j_2} \dots (a^\dagger_{(L)} - a^\dagger_{(L-1)})^{j_{L-1}}\,|0\rangle\,, \tag{2.7}$$

with spin $J_3 = \frac{1}{2} + \sum_s j_s$. In particular for $L = 2$ one finds the known result

$$V_F \times V_F = \sum_{j=0}^{\infty} V_j\,, \tag{2.8}$$

with V_j generated by acting with J_+ on the HWS $|j\rangle = (a^\dagger_{(2)} - a^\dagger_{(1)})^j\,|0\rangle$. The corresponding states in free $\mathcal{N} = 4$ SYM follow from the dictionary

$$(a^\dagger_{(1)})^{n_1}\,(a^\dagger_{(2)})^{n_2} \dots (a^\dagger_{(L)})^{n_L}\,|0\rangle \quad \leftrightarrow \quad \mathcal{D}^{n_1} Z\, \mathcal{D}^{n_2} Z \dots \mathcal{D}^{n_L} Z\,, \tag{2.9}$$

with $n_i \geq 0$.

$\mathfrak{hs}(1,1)$. The HS extension $\mathfrak{hs}(1,1)$ of $\mathfrak{sl}(2)$ is defined by introducing the HS generators [25]

$$J_{p,q} = (a^\dagger)^p a^q\,. \tag{2.10}$$

The $J_{p,q}$ clearly close under the commutator / product into an HS algebra that contains the $\mathfrak{sl}(2)$ subalgebra (2.3). We call it $\mathfrak{hs}(1,1)$. The generators $J_{p,q}$ with $p < q$ are raising operators. The singleton is again a representation of this algebra since $|0\rangle$ is annihilated by all raising operators. In the tensor product of L singletons, HWS's of $\mathfrak{hs}(1,1)$ are solutions of

$$\sum_{i=1}^{L} (a^\dagger_{(i)})^p\, \partial_i^q\, f(a^\dagger_{(1)}, \dots, a^\dagger_{(L)}) = 0 \quad \text{with } p < q\,. \tag{2.11}$$

Equations (2.11) are highly restrictive and solutions are rare. Our claim is that solutions to these equations, i.e. irreducible representations of $\mathfrak{hs}(1,1)$, are in one-to-one correspondence with Young tableaux (YT) made out of L boxes, i.e. row increasing diagrams with boxes numbered always increasingly along rows and columns (for a quick review on YT decompositions see [77, appendix]).

Since $\mathfrak{hs}(1,1)$ HWS's are also HWS's of its $\mathfrak{sl}(2)$ subalgebra we can restrict our attention to states of the type (2.7). For instance, for $L=2$, the condition $J_{0,1} f(a^\dagger_{(1)}, a^\dagger_{(2)})|0\rangle = 0$ is solved by

$$|j\rangle = (a^\dagger_{(2)} - a^\dagger_{(1)})^j |0\rangle \,. \tag{2.12}$$

The conditions $J_{0,n}|j\rangle = 0$ leave only $|0\rangle$ and $|1\rangle$ as solutions. Indeed the two states automatically satisfy $J_{m\geq 1, p\geq 2}|j\rangle = 0$ and therefore are HWS's. They correspond to the two HWS's in the symmetric and antisymmetric tensor product of two singletons respectively

$$\begin{aligned} \begin{array}{|c|c|}\hline 1&2\\\hline\end{array} &= |0\rangle \quad \leftrightarrow \quad Z^2\,, \\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline\end{array} &= \left(a^\dagger_{(2)} - a^\dagger_{(1)}\right)|0\rangle \quad \leftrightarrow \quad Z(\mathcal{D}Z) - (\mathcal{D}Z)Z\,. \end{aligned} \tag{2.13}$$

In $\mathcal{N}=4$ SYM, the antisymmetric doubleton is projected out after tracing over gauge indices. The generalization to $L>2$ states is straightforward. The HWS for the completely symmetric representation is again given by $f(a^\dagger_{(1)}, \ldots, a^\dagger_{(L)}) = 1$ while the HWS for the completely antisymmetric tableau can be written as a product of all the differences

$$\begin{array}{|c|}\hline \\\hline \\\hline \\\hline \\\hline\end{array} = \prod_{i>j}^{L}\left(a^\dagger_{(i)} - a^\dagger_{(j)}\right)|0\rangle \quad \leftrightarrow \quad Z\,(\mathcal{D}Z)\,(\mathcal{D}^2 Z)\ldots(\mathcal{D}^{L-1}Z) + \text{antisymm.} \tag{2.14}$$

That this state satisfies (2.11) can be seen by noticing that being completely antisymmetric, derivatives $\sum_i (a^\dagger_{(i)})^p\,\partial_i^q$ in the completely symmetric operator $J_{p,q}$ cancel against each other. Similarly one can build more general solutions from tensoring k columns of type (2.14) leading to

$$\begin{array}{|c|c|c|c|}\hline &&&\\\hline &&\multicolumn{2}{c}{}\\\cline{1-2} &\multicolumn{3}{c}{}\\\cline{1-1} &\multicolumn{3}{c}{}\\\cline{1-1}\end{array} = \prod_{p=1}^{k}\prod_{i_p>j_p}^{L_p}(a^\dagger_{(i_p)} - a^\dagger_{(j_p)})|0\rangle \quad \leftrightarrow \quad Z^{n_1}\,(\mathcal{D}Z)^{n_2}\ldots(\mathcal{D}^{n_s}Z) + \text{perms.} \tag{2.15}$$

with $L = \sum_p L_p$, n_i the number of boxes in the i^{th} row and "perms" denoting all permutations specified by the tableau. To each of these solutions we associate a Young Tableaux with k columns of length L_p and boxes labelled by $i_p \in \{1,2,\ldots,L_p\}$. We believe that these are the only solutions to (2.11) but we have no rigorous proof of this uniqueness.

For example, HWS's for $L=3$ are given by

$$\begin{aligned} \begin{array}{|c|c|c|}\hline 1&2&3\\\hline\end{array} &= |0\rangle \quad \leftrightarrow \quad Z^3\,, \\ \begin{array}{|c|c|}\hline 1&2\\\hline 3&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} &= \left(a^\dagger_{(3)} - a^\dagger_{(1)}\right)|0\rangle \quad \leftrightarrow \quad Z^2(\mathcal{D}Z) - (\mathcal{D}Z)Z^2\,, \\ \begin{array}{|c|c|}\hline 1&3\\\hline 2&\multicolumn{1}{c}{}\\\cline{1-1}\end{array} &= \left(a^\dagger_{(2)} - a^\dagger_{(1)}\right)|0\rangle \quad \leftrightarrow \quad Z(\mathcal{D}Z)Z - (\mathcal{D}Z)Z^2\,, \\ \begin{array}{|c|}\hline 1\\\hline 2\\\hline 3\\\hline\end{array} &= \left(a^\dagger_{(2)} - a^\dagger_{(1)}\right)\left(a^\dagger_{(3)} - a^\dagger_{(1)}\right)\left(a^\dagger_{(3)} - a^\dagger_{(2)}\right)|0\rangle \quad \leftrightarrow \quad Z(\mathcal{D}Z)(\mathcal{D}^2 Z) + \text{antisymm.} \end{aligned} \tag{2.16}$$

In $\mathcal{N}=4$ SYM, the two HS 3-pleton multiplets associated to the hooked tableaux $\begin{array}{|c|c|}\hline &\\\hline &\multicolumn{1}{c}{}\\\cline{1-1}\end{array}$ are projected out after tracing over the gauge indices.

2.2 A first look at $\mathfrak{hs}(2,2|4)$

In order to extend the previous analysis to the case of our main interest, the higher spin algebra $\mathfrak{hs}(2,2|4)$, we need to recall some basic properties of this infinite dimensional HS (super)algebra. To this end we closely follow [34] and adopt their notations with minor changes. The $\mathcal{N}=4$ superconformal algebra $\mathfrak{psu}(2,2|4)$ can be realized in terms of (super-) oscillators $\zeta_\Lambda = (y_a, \theta_A)$ with:

$$[y_a, \bar{y}^b] = \delta_a{}^b\,, \qquad \{\theta_A, \bar{\theta}^B\} = \delta^B{}_A\,, \tag{2.17}$$

where $y_a, \bar{y}^b$ are bosonic oscillators with $a, b = 1, \ldots 4$ a Weyl spinor index of $\mathfrak{so}(4,2) \sim \mathfrak{su}(2,2)$ or, equivalently, a Dirac spinor index of $\mathfrak{so}(4,1)$, while $\theta_A, \bar{\theta}^B$ are fermionic oscillators with $A, B = 1, \ldots, 4$ a Weyl spinor index of $\mathfrak{so}(6) \sim \mathfrak{su}(4)$.

Generators of $\mathfrak{psu}(2,2|4)$ are written as 'traceless' bilinears $\bar{\zeta}^\Sigma \zeta_\Lambda$ of superoscillators. In particular, the 'diagonal' combinations realize the compact $\mathfrak{so}(6)$ and noncompact $\mathfrak{so}(4,2)$ bosonic subalgebras respectively, while the mixed combinations generate supersymmetries:

$$\begin{aligned} J^a{}_b &= \bar{y}^a y_b - \frac{1}{2} K \delta^a{}_b\,, & K &= \frac{1}{2} \bar{y}^a y_a\,, \\ T^A{}_B &= \bar{\theta}^A \theta_B - \frac{1}{2} B \delta^A_B\,, & B &= \frac{1}{2} \bar{\theta}^A \theta_A\,, \\ \mathcal{Q}^A_a &= \bar{\theta}^A y_a\,, & \bar{\mathcal{Q}}^a_A &= \bar{y}^a \theta_A\,. \end{aligned} \tag{2.18}$$

The combination

$$C \equiv K + B = \frac{1}{2} \bar{\zeta}^\Lambda \zeta_\Lambda\,, \tag{2.19}$$

commutes with all the remaining generators and is thus a central element. The abelian ideal generated by C can be modded out e.g. by setting C to zero. At least in perturbation theory, this should make physical sense, since the elementary SYM fields $\{A_\mu, \lambda^\alpha_A, \bar{\lambda}^A_{\dot{\alpha}}, \varphi^i\}$ and their composites all have $C = 0$.[6] Finally, the combination B is to be identified as the generator of Intriligator's "bonus symmetry" [90, 91] dual to the 'anomalous' $\mathrm{U}(1)_B$ chiral symmetry of type IIB in the AdS bulk. It acts as an external automorphism [34] that rotates the supercharges of the SCA. The $\mathfrak{psu}(2,2|4)$ invariant vacuum $|0\rangle$, annihilated by ζ_Λ, corresponds to the identity operator which can be viewed as the trivial singlet representation.

The HS extension $\mathfrak{hs}(2,2|4)$ is roughly speaking generated by odd powers of the above generators i.e combinations with equal odd numbers of ζ_Λ and $\bar{\zeta}^\Lambda$. More precisely, one first considers the enveloping algebra of $\mathfrak{psu}(2,2|4)$, which is an associative algebra and consists of all powers of the generators, then restricts it to the odd part which closes as a Lie algebra modulo the central charge C, and finally quotients the ideal generated by C. It is easy to show that B is never generated in commutators (but C is!) and thus remains

[6] In principle, one can consider quotienting by $C - C_0$, where C_0 is any (half) integer. This would correspond to choosing as the basic building block some singleton of $\mathrm{SU}(2,2|4)$ with non vanishing central charge $C = C_0$. These non self-conjugate singletons play only a marginal accessory role in (perturbative) $\mathcal{N}=4$ SYM theory [85]–[88] and [34, 35, 15, 89].

an external automorphism of $\mathfrak{hs}(2,2|4)$. Generators of $\mathfrak{hs}(2,2|4)$ can be represented by 'traceless' polynomials in the superoscillators:

$$\mathfrak{hs}(2,2|4) = \oplus_\ell \mathcal{A}_{2\ell+1} = \sum_{\ell=0}^{\infty} \left\{ \mathcal{J}_{2\ell+1} = P^{\Lambda_1 \ldots \Lambda_{2\ell+1}}_{\Sigma_1 \ldots \Sigma_{2\ell+1}} \bar{\zeta}^{\Sigma_1} \ldots \bar{\zeta}^{\Sigma_{2\ell+1}} \zeta_{\Lambda_1} \ldots \zeta_{\Lambda_{2\ell+1}} \right\}, \tag{2.20}$$

with elements $\mathcal{J}_{2\ell+1}$ in $\mathcal{A}_{2\ell+1}$, where ℓ is called the level, parametrized by traceless rank $(2\ell+1)$ (graded) symmetric tensors $P^{\Lambda_1 \ldots \Lambda_{2\ell+1}}_{\Sigma_1 \ldots \Sigma_{2\ell+1}}$. The commutators of two elements however close only up to the ideal generated by C. In particular they close on the subspace of physical states defined by the condition $C \equiv 0$. The restriction to this subspace will be always understood. Alternatively, the HS algebra can be more generally defined by identifying generators differing by terms that involve C, i.e. $\mathcal{J} \approx \mathcal{K}$ iff $\mathcal{J} - \mathcal{K} = \sum_{k \geq 1} C^k \mathcal{H}_k$ [34].

To each element in $\mathcal{A}_{2\ell+1}$ with $\mathfrak{su}(2)_L \times \mathfrak{su}(2)_R$ spins $[j, \bar{\jmath}]$ is associated an $\mathfrak{hs}(2,2|4)$ HS gauge field in the AdS bulk with labels $[j+\frac{1}{2}, \bar{\jmath}+\frac{1}{2}]$. The $\mathfrak{su}(4) \times \mathfrak{su}(2)^2$ content of the HS currents can be easily read off from (2.20) by expanding the polynomials in powers of θ's up to 4, since $\theta^5 = 0$. There is a single superconformal multiplet $\mathcal{V}_{2\ell}$ at each level $\ell \geq 2$. The lowest spin cases $\ell = 0, 1$, i.e. $\hat{\mathcal{V}}_{0,2}$, are special. They differ from the content of doubleton multiplets $\mathcal{V}_{0,2}$ by spin $s < 1$ states [34]. The content of (2.20) can then be written as ([34, tables 4,5])

$$\begin{aligned}
\hat{\mathcal{V}}_0 &= \left| \bar{\mathbf{4}}_{[\frac{1}{2},0]} + \mathbf{1}_{[1,0]} \right|^2 - \mathbf{1}_{[\frac{1}{2},\frac{1}{2}]} \\
\hat{\mathcal{V}}_2 &= \left| \mathbf{4}_{[\frac{1}{2},0]} + \mathbf{6}_{[1,0]} + \bar{\mathbf{4}}_{[\frac{3}{2},0]} + \mathbf{1}_{[2,0]} \right|^2 \\
\mathcal{V}_{2\ell} &= \left| \mathbf{1}_{[\ell-1,0]} + \mathbf{4}_{[\ell-\frac{1}{2},0]} + \mathbf{6}_{[\ell,0]} + \bar{\mathbf{4}}_{[\ell+\frac{1}{2},0]} + \mathbf{1}_{[\ell+1,0]} \right|^2, \qquad \ell \geq 2,
\end{aligned} \tag{2.21}$$

with $\mathbf{r}_{[j+\frac{1}{2}, \bar{\jmath}+\frac{1}{2}]}$ denoting the $\mathfrak{su}(4)$ representation $\mathbf{r}$ and the labels of the $\mathfrak{u}(1)^2 \in \mathfrak{su}(2)^2$ HWS's. Complex conjugates are given by conjugating $\mathfrak{su}(4)$ representations and exchanging the spins $j \leftrightarrow \bar{\jmath}$. The product is understood in $\mathfrak{su}(4)$ while $\mathfrak{u}(1)^2$ labels simply add. The highest spin state $\mathbf{1}_{[\ell+1,\ell+1]}$ corresponds to the state $y^{2\ell+1}\bar{y}^{2\ell+1}$ with no θ's, $\mathbf{4}_{[\ell+\frac{1}{2},\ell+1]}$, $\bar{\mathbf{4}}_{[\ell+1,\ell+\frac{1}{2}]}$ to $y^{2\ell}\bar{y}^{2\ell+1}\theta^A$, $y^{2\ell+1}\bar{y}^{2\ell}\bar{\theta}_A$, and so on. For $\ell = 0, 1$, states with negative $j, \bar{\jmath}$ should be deleted. In addition we subtract the current $\mathbf{1}_{[\frac{1}{2},\frac{1}{2}]}$ at $\ell = 0$ associated to C. In the $\mathcal{N} = 4$ notation introduced in appendix A, $\mathcal{V}_{2\ell}$ corresponds to the semishort multiplet $\mathcal{V}^{2\ell,0}_{[000][\ell-1^*,\ell-1^*]}$ (see also table 4 in appendix C).

2.3 Representations of $\mathfrak{hs}(2,2|4)$

The basic representation of both $\mathfrak{psu}(2,2|4)$ and $\mathfrak{hs}(2,2|4)$ is the so called "singleton" $\mathcal{V}^{1,0}_{[0,1,0][0,0]}$ associated to the $\mathcal{N} = 4$ SYM vector multiplet. Its HWS $|Z\rangle$, i.e. the ground-state or 'vacuum', which is obviously different from the trivial $\mathfrak{psu}(2,2|4)$ invariant vacuum $|0\rangle$, is one of the complex scalars, let us say $Z = \varphi^5 + i\varphi^6$. The other (complex) components will be denoted by $X = \varphi^1 + i\varphi^2$ and $Y = \varphi^3 + i\varphi^4$ in the following. Showing that the singleton is an irreducible representation of $\mathfrak{psu}(2,2|4)$ is tantamount to showing that any elementary SYM state can be found by acting on the Fock space vacuum $|Z\rangle$ with a sequence of superconformal generators chosen among (2.18). Looking at the singleton as an irrep of

$\mathfrak{hs}(2,2|4)$ one sees an important difference: the sequence of superconformal generators[7] is replaced by a single HS generator and therefore any component A in the singleton multiplet can be reached in a single step $\mathcal{J}_{A\bar{B}}$ from any other one B. This can be shown by noticing that, since the central charge C commutes with all generators and annihilates the vacuum, a non-trivial sequence in $(\mathcal{A}_1)^{2\ell+1}$ belongs to $\mathcal{A}_{2\ell+1}$. This difference, irrelevant for one-letter states ($L=1$), will be crucial in proving the irreducibility of YT-pletons with respect to the HS algebra.[8]

Let us now consider the tensor product of L singletons. The generators of $\mathfrak{hs}(2,2|4)$ are realized as diagonal combinations:

$$\mathcal{J}_{2\ell+1} \equiv \sum_{s=1}^{L} \mathcal{J}_{2\ell+1}^{(s)} \tag{2.22}$$

with $\mathcal{J}_{2\ell+1}^{(s)}$ HS generators acting at the s^{th} site. The tensor product of $L \geq 1$ singletons is generically reducible not only under $\mathfrak{psu}(2,2|4)$ but also under $\mathfrak{hs}(2,2|4)$. This can be seen by noticing that the HS generators (2.22), being completely symmetric, commute with symmetrizations and antisymmetrizations of the indices in the tensor product of singletons. In particular, the tensor product decomposes into a sum of representations characterized by Young tableaux YT with L boxes. A Young tableaux is defined by distributing SYM letters among L boxes and acting on it with the operator $\mathcal{O}_{\text{YT}} = A_{\text{YT}} S_{\text{YT}}$ that first symmetrizes all letters in the same row and then antisymmetrizes letters in the same column. This operator clearly commutes with all generators of $\mathfrak{hs}(2,2|4)$, and therefore different Young tableaux belong to different irreducible components.

To prove irreducibility of L-pletons associated to a specific YT with L boxes under $\mathfrak{hs}(2,2|4)$, it is then enough to show that any state in the L-pleton under consideration can be found by acting on the relevant HWS with HS generators. Let us start by considering states belonging to the totally symmetric tableau. The simplest examples of such states are those with only *one* site different from the vacuum Z, i.e. $AZ\ldots Z+\text{symm.}$. Using the fact that any SYM letter A can reached from the HWS Z using a single $\mathfrak{hs}(2,2|4)$ generator $\mathcal{J}_{A\bar{Z}}$ we write the "one impurity" state as $(\mathcal{J}_{A\bar{Z}}Z)Z\ldots Z+\text{symm}$. This state can also be written as $\mathcal{J}_{A\bar{Z}}(Z^L)$ and it is therefore a HS descendant. The next simplest class is given by states with "two impurities" $ABZ\ldots Z+\text{symm.}$. Once again this state can be written as $\mathcal{J}_{A\bar{Z}}\mathcal{J}_{B\bar{Z}}(Z^L)$ up to the "one impurity" descendant $(\mathcal{J}_{A\bar{Z}}\mathcal{J}_{B\bar{Z}}Z)Z\ldots Z$ of the type already found. Proceeding in this way the reader can easily convince him/herself that all states in the completely symmetric tensor of L singletons can be written as HS descendants of the vacuum Z^L.

The same arguments hold for generic tableaux. For example, besides the descendants $\mathcal{J}_{A\bar{Z}}(Z^L)$ of Z^L there are $L-1$ "one impurity" multiplets of states associated to the $L-1$

[7]Without loss of generality we may assume the length of the sequence to be odd; for an even sequence we may append an element of the Cartan subalgebra, e.g. the dilatation generator.

[8]This property is also satisfied by the fundamental representation of $\mathrm{SU}(m)$. Our proof below reduces in this case to the familiar statement that irreducible representations of $\mathrm{SU}(m)$ are in one-to-one correspondence with Young tableaux made out of fundamentals.

Young tableaux with $L-1$ boxes in the first row and a single box in the second one.[9] The vacuum state of HS multiplets associated to such tableaux can be taken to be $Y_{(k)} \equiv Z^k Y Z^{L-k-1} - Y Z^{L-1}$ with $k = 1, \dots, L-1$. Any state with one impurity $Z^k A Z^{L-k-1} - A Z^{L-1}$ with $k = 1, \dots, L-1$ can be found by acting on $Y_{(k)}$ with the HS generator $\mathcal{J}_{A\bar{Y}}$, where $\mathcal{J}_{A\bar{Y}}$ is the HS generator that transforms Y into A (and annihilates Z).

Notice that the arguments rely heavily on the fact that any two states in the singleton are related by a one-step action of a HS generator. This is not the case for the $\mathcal{N}=4$ SCA, and indeed the completely symmetric tensor product of L singletons is highly reducible with respect to $\mathfrak{psu}(2,2|4)$, as we shall see in the following.

3. HS content of $\mathcal{N}=4$ SYM

The on-shell field content of the singleton representation of $\mathfrak{psu}(2,2|4)$ is encoded in the partition function

$$\mathcal{Z}_{\Box}(t, y_i) = \sum_{s=0}^{\infty} \Big[t^{1+s} \chi_{[\frac{s}{2},\frac{s}{2}]} \chi_{[010]} + t^{2+s} \chi_{[\frac{s+2}{2},\frac{s}{2}]} \chi_{[000]} + t^{2+s} \chi_{[\frac{s}{2},\frac{s+2}{2}]} \chi_{[000]} + \\ - t^{\frac{3+s}{2}} \chi_{[\frac{s+1}{2},\frac{s}{2}]} \chi_{[001]} - t^{\frac{3+s}{2}} \chi_{[\frac{s}{2},\frac{s+1}{2}]} \chi_{[100]} \Big], \tag{3.1}$$

with the different terms corresponding to the six real scalars φ^i, the field strengths $F^{\pm}_{\mu\nu}$ and the fermions λ^{α}_{A}, $\bar{\lambda}^{A}_{\dot{\alpha}}$, respectively, together with their derivatives. Here t keeps track of the bare conformal dimension Δ. $\chi_{[j,\bar{\jmath}]}\chi_{[q_1,p,q_2]}(y_i)$ denotes the character polynomial of the $\mathfrak{so}(4) \times \mathfrak{so}(6)$ representation $[j,\bar{\jmath}][q_1,p,q_2]$.[10] In particular, focusing only on the scaling dimensions Δ and performing explicitly the sum over s, one finds the one-letter partition function[11]

$$\mathcal{Z}_{\Box}(t, y_i)|_{y_i=1} = \frac{2\,t\,(3 + t^{\frac{1}{2}})}{(1 + t^{\frac{1}{2}})^3}\,. \tag{3.2}$$

As explained above, the singleton turns out to be the "fundamental representation" of $\mathfrak{hs}(2,2|4)$ as well. Moreover, we have argued that representations of $\mathfrak{hs}(2,2|4)$ are built in terms of tensor products of singletons properly decomposed according to irreducible representations of the permutation group. These are associated to Young Tableaux built from $\Box \equiv \mathcal{Z}_{\Box}(t)$. The spectrum of single-trace operators in $\mathcal{N}=4$ SYM theory with SU(N) gauge group is given by all possible *cyclic* words built from letters chosen from $\mathcal{Z}_{\Box}$. It can be computed using Polya theory [93], which gives the generating function [2]–[4] and [7, 29]

$$\mathcal{Z}(u, t, y_i) = \sum_{n \geq 2} u^n\, \mathcal{Z}_n(t, y_i) = \sum_{n \geq 2, d|n} u^n\, \frac{\varphi(d)}{n}\, \mathcal{Z}_{\Box}(t^d, y_i^d)^{\frac{n}{d}}\,, \tag{3.3}$$

[9] As we will momentarily see, HS multiplets of this kind are absent for $\mathcal{N}=4$ SYM theories with semisimple gauge group. At any rate, they are instrumental to illustrate our point.

[10] Characters of UIR's of superconformal algebras have been recently derived in [92].

[11] At $y_i = 1$ one has by definition $\chi_{[q_1,p,q_2]} = \dim[q_1,p,q_2] = (q_1+1)(p+1)(q_2+1)(p+q_1+2)(p+q_2+2)(p+q_1+q_2+3)/12$ and $\chi_{[j,\bar{\jmath}]} = \dim[j,\bar{\jmath}] = (2j+1)(2\bar{\jmath}+1)$.

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for cyclic words. Here u keeps track of the length L, i.e. the number of letters / partons. The sum runs over all integers $n > 2$ and their divisors d, and $\varphi(d)$ is Euler's totient function, that equals the number of integers smaller than and relatively prime to d. For later convenience, we have introduced the notation $\mathcal{Z}_n(t, y_i)$ to denote the restriction to cyclic words made out of n-letters. The partition function (3.3) accounts for SYM composite operators and all their derivatives, i.e. their $\mathfrak{so}(4,2)/(\mathfrak{so}(4)\times\mathfrak{so}(2))$ descendants. $\mathfrak{so}(4,2)$ primaries can instead be read off from $\widehat{\mathcal{Z}}(u,t,y_i)$, defined from $\mathcal{Z}(u,t,y_i)$ by removing total derivatives:

$$\widehat{\mathcal{Z}}(u,t,y_i) \equiv \mathcal{Z}(u,t,y_i)\left(1 - t\,\chi_{[\frac{1}{2}\frac{1}{2}]} + t^2\left(\chi_{[10]} + \chi_{[01]}\right) - t^3\,\chi_{[\frac{1}{2}\frac{1}{2}]} + t^4\right). \tag{3.4}$$

We note that $\mathcal{Z}_{\ydiagram{1}}(u^d, t^d, y_i^d)$ denotes the alternating sum over length-d Young tableaux of the hook type:

$$\mathcal{Z}_{\ydiagram{1}}(t^d) = \mathcal{Z}_{\ydiagram{5}\cdots\ydiagram{2}}(t) - \mathcal{Z}_{\ydiagram{4,1}\cdots\ydiagram{2}}(t) + \mathcal{Z}_{\ydiagram{3,1,1}\cdots\ydiagram{2}}(t) - \mathcal{Z}_{\ydiagram{2,1,1,1}\cdots\ydiagram{2}}(t) + \cdots. \tag{3.5}$$

Plugging this expansion into (3.3), we find for the first few cases:

$$\begin{aligned}
\mathcal{Z}_2 &= \mathcal{Z}_{\ydiagram{2}}, \\
\mathcal{Z}_3 &= \mathcal{Z}_{\ydiagram{3}} + \mathcal{Z}_{\ydiagram{1,1,1}}, \\
\mathcal{Z}_4 &= \mathcal{Z}_{\ydiagram{4}} + \mathcal{Z}_{\ydiagram{2,1,1}} + \mathcal{Z}_{\ydiagram{2,2}}, \\
\mathcal{Z}_5 &= \mathcal{Z}_{\ydiagram{5}} + \mathcal{Z}_{\ydiagram{3,2}} + 2\,\mathcal{Z}_{\ydiagram{3,1,1}} + \mathcal{Z}_{\ydiagram{2,2,1}} + \mathcal{Z}_{\ydiagram{1,1,1,1,1}}, \qquad \text{etc.}
\end{aligned} \tag{3.6}$$

Notice that only a subset of YT, those compatible with cyclicity of the trace, enters in (3.6). In particular, HS multiplets associated to the tableaux $\ydiagram{1,1}$, $\ydiagram{2,1}$, two out of the three of type $\ydiagram{2,1,1}$, and so on, are projected out. The content of the various components in (3.6) can be derived from the formulae:

$$\begin{aligned}
\mathcal{Z}_{\ydiagram{2}} &= \frac{1}{2!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^2 + \mathcal{Z}_{\ydiagram{1}}(t^2)\right] \\
\mathcal{Z}_{\ydiagram{3}} &= \frac{1}{3!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^3 + 3\,\mathcal{Z}_{\ydiagram{1}}(t^2)\mathcal{Z}_{\ydiagram{1}}(t) + 2\,\mathcal{Z}_{\ydiagram{1}}(t^3)\right] \\
\mathcal{Z}_{\ydiagram{1,1,1}} &= \frac{1}{3!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^3 - 3\,\mathcal{Z}_{\ydiagram{1}}(t^2)\mathcal{Z}_{\ydiagram{1}}(t) + 2\,\mathcal{Z}_{\ydiagram{1}}(t^3)\right] \\
\mathcal{Z}_{\ydiagram{4}} &= \frac{1}{4!}\left[\mathcal{Z}_{\ydiagram{1}}(t)^4 + 6\,\mathcal{Z}_{\ydiagram{1}}(t^2)\mathcal{Z}_{\ydiagram{1}}(t)^2 + 3\,\mathcal{Z}_{\ydiagram{1}}(t^2)^2 + 8\,\mathcal{Z}_{\ydiagram{1}}(t^3)\mathcal{Z}_{\ydiagram{1}}(t) + 6\,\mathcal{Z}_{\ydiagram{1}}(t^4)\right] \\
\mathcal{Z}_{\ydiagram{2,2}} &= \frac{1}{4!}\left[2\,\mathcal{Z}_{\ydiagram{1}}(t)^4 + 6\,\mathcal{Z}_{\ydiagram{1}}(t^2)^2 - 8\,\mathcal{Z}_{\ydiagram{1}}(t^3)\,\mathcal{Z}_{\ydiagram{1}}(t)\right] \\
\mathcal{Z}_{\ydiagram{2,1,1}} &= \frac{1}{4!}\left[3\,\mathcal{Z}_{\ydiagram{1}}(t)^4 - 6\,\mathcal{Z}_{\ydiagram{1}}(t^2)\mathcal{Z}_{\ydiagram{1}}(t)^2 - 3\,\mathcal{Z}_{\ydiagram{1}}(t^2)^2 + 6\,\mathcal{Z}_{\ydiagram{1}}(t^4)\right].
\end{aligned} \tag{3.7}$$

Formulae (3.7) can be explicitly verified with the use of (3.5).

Under the superconformal group $\mathfrak{psu}(2,2|4)$, the HS multiplet $\mathcal{Z}_{YT}$, associated to a given Young tableau YT with L boxes, decomposes into an infinite sums of multiplets.

L	name	$\mathcal{V}^{\Delta,B}_{[j,\bar{j}][q_1,p,q_2]}$	sector
2	$\mathcal{V}_0$	$\mathcal{V}^{2,0}_{[0^\dagger,0^\dagger][0,2,0]}$	$\mathfrak{sl}(2)_{j=-\frac{1}{2}}$
2	$\mathcal{V}_n$	$\mathcal{V}^{n,0}_{[\frac{1}{2}n-1^*,\frac{1}{2}n-1^*][0,0,0]}$	$\mathfrak{sl}(2)_{j=-\frac{1}{2}}$
3	$\mathcal{V}_{0,0}$	$\mathcal{V}^{3,0}_{[0^\dagger,0^\dagger][0,3,0]}$	$\mathfrak{sl}(2)_{j=-\frac{1}{2}}$
3	$\mathcal{V}_{0,n}$	$\mathcal{V}^{n+1,0}_{[\frac{1}{2}n-1^*,\frac{1}{2}n-1^*][0,1,0]}$	$\mathfrak{sl}(2)_{j=-\frac{1}{2}}$
3	$\mathcal{V}_{1,n}$	$\mathcal{V}^{n+\frac{5}{2},+\frac{1}{2}}_{[n/2^*,n/2-1/2^*][0,0,1]}$	$\mathfrak{sl}(2)_{j=-1}$
3	$\mathcal{V}_{-1,n}$	$\mathcal{V}^{n+\frac{5}{2},-\frac{1}{2}}_{[n/2-1/2^*,n/2^*][1,0,0]}$	$\mathfrak{sl}(2)_{j=-1}$
3	$\mathcal{V}_{m\geq+2,n}$	$\mathcal{V}^{n+2m,1}_{[\frac{1}{2}n+m-1^*,\frac{1}{2}n][0,0,0]}$	$\mathfrak{su}(1,2)$
3	$\mathcal{V}_{m\leq-2,n}$	$\mathcal{V}^{n+2m,-1}_{[\frac{1}{2}n,\frac{1}{2}n+m-1^*][0,0,0]}$	$\mathfrak{su}(1,2)$

Table 1: $\mathfrak{psu}(2,2|4)$ multiplets with $L \leq 3$.

The HWS's can be found by computing $\mathcal{Z}_{YT}$ and eliminating the superconformal descendants by passing $\mathcal{Z}_{YT}$ through a sort of Erathostenes' (super) sieve [29]. This will be the subject of the next subsection. Here we just state the results for $L = 2, 3$. The complete list of $\mathfrak{psu}(2,2|4)$ multiplets appearing in the decomposition of the first few HS multiplets with $L = 2, 3$ letters is collected in table 1, see appendix A for the notation of $\mathfrak{psu}(2,2|4)$ multiplets. The decompositions of the corresponding HS multiplets reads:

$$
\begin{aligned}
\mathcal{Z}_{\yng(2)} &= \sum_{n=0}^{\infty} \mathcal{V}_{2n}\,, \qquad \mathcal{Z}_{\yng(1,1)} = \sum_{n=0}^{\infty} \mathcal{V}_{2n+1}\,, \\
\mathcal{Z}_{\yng(3)} &= \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} c_n \left[\mathcal{V}_{2k,n} + \mathcal{V}_{2k+1,n+3}\right], \\
\mathcal{Z}_{\yng(2,1)} &= \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} d_n \left[\mathcal{V}_{2k,n+1} + \mathcal{V}_{2k+1,n+1}\right], \\
\mathcal{Z}_{\yng(1,1,1)} &= \sum_{k=-\infty}^{\infty} \sum_{n=0}^{\infty} c_n \left[\mathcal{V}_{2k,n+3} + \mathcal{V}_{2k+1,n}\right].
\end{aligned} \tag{3.8}
$$

The coefficients $c_n \equiv 1 + [n/6] - \delta_{n,1 \bmod 6}$ and $d_n \equiv 1 + [n/3]$ with $[m]$ the integral part of m, are the multiplicities of $\mathfrak{psu}(2,2|4)$ multiplets inside $\mathfrak{hs}(2,2|4)$. More precisely c_n, d_n count the number of ways one can distribute derivatives (HS descendants) between the boxes in the tableaux. These multiplicities will be computed in the next section, cf. (4.23) below. For convenience of the reader we display the translation of these formulae into $\mathfrak{psu}(2,2|4)$ notation $\mathcal{V}^{\Delta,B}_{[j,\bar{j}][q_1,p,q_2]}$ in appendix C.

The multiplets with $n = 0$ or $m = 0, \pm 1$ in table 1 are special: $n = 0$ corresponds to the $\frac{1}{2}$-BPS series, dual to $\mathcal{N} = 8$ gauged supergravity and its KK recurrences, $m = 0, \pm 1, n \geq 1$ to semishort-semishort multiplets. Finally for $m \geq 2$ one finds multiplets satisfying a bound of type long-semishort.

The ‘symmetric doubleton’ $\mathcal{Z}_{\square\square}$ contains the multiplets of conserved HS currents $\mathcal{V}_{2n}$. The ‘antisymmetric doubleton’ $\mathcal{Z}_{\begin{smallmatrix}\square\\\square\end{smallmatrix}}$ is ruled out by cyclicity of the trace, cf. (3.6). The ‘symmetric tripleton’ $\mathcal{Z}_{\square\square\square}$ (corresponding to the cubic Casimir d_{abc}) contains the first KK recurrences of twist 2 semishort multiplets, the still semishort-semishort series $\mathcal{V}_{\pm 1,n}$ starting with fermionic primaries and long-semishort multiplets. The ‘antisymmetric tripleton’ $\mathcal{Z}_{\begin{smallmatrix}\square\\\square\\\square\end{smallmatrix}}$ (corresponding to the structure constants f_{abc}) on the other hand contains the Goldstone multiplets that join to multiplets with twist 2 to form long multiplets when the HS symmetry is broken. In addition, fermionic semishort-semishort multiplets and long-semishort multiplets also appear.

4. Partition function of semishort superprimaries

In this section, we focus on the particularly interesting class of SYM operators sitting in BPS and semishort multiplets of the superconformal algebra $\mathfrak{psu}(2,2|4)$ and derive multiplicity formulae for their superprimaries. Semishort and BPS multiplets are special in that their components encompass all generalized ‘massless’ states and their superpartners. By this we mean SYM operators whose dimensions saturate unitary bounds and whose holographic duals would thus be massless in a manifestly SO(10, 2) symmetric description in the bulk [94, 31, 95]. Not unexpectedly, we will find that general formulae drastically simplify for these operators. When interactions are turned on ($g_{YM} \neq 0$), i.e. departing from the HS enhancement point, only truly 1/2 BPS multiplets remain ‘massless’ in the above generalized sense. All semishort multiplets participate in the ‘Grande Bouffe’, whereby they ‘eat’ the relevant Goldstone / Stückelberg multiplets and become massive. The resulting long multiplets acquire anomalous dimensions and, in principle, mix with one another compatibly with their quantum numbers.

A generic long $\mathfrak{psu}(2,2|4)$ multiplet will be denoted as $\mathcal{V}^{\Delta,B}_{[q_1,p,q_2][j,\bar{j}]}$ by means of the Dynkin labels of its HWS with respect to the compact bosonic subalgebra $\mathfrak{su}(4)\times\mathfrak{su}(2)^2\times\mathfrak{u}(1)_\Delta$ and the ‘external’ $\mathfrak{u}(1)_B$ hypercharge. More precisely, $[q_1,p,q_2]$ are Dynkin labels of $\mathfrak{su}(4)$ while $[j,\bar{j}]$ denote the spins under $\mathfrak{su}(2)\times\mathfrak{su}(2)$. At particular values of Δ, the long multiplet $\mathcal{V}^{\Delta,B}_{[q_1,p,q_2][j,\bar{j}]}$ may split into semi-short or BPS multiplets, cf. appendix A for details.

For the following, it is convenient to split superoscillator indices with respect to the $\mathfrak{su}(2)_a\times\mathfrak{su}(2)_b\times\mathfrak{su}(2)_c\times\mathfrak{su}(2)_d$ subalgebra inside $\mathfrak{su}(2,2)\times\mathfrak{su}(4)$, which yields $y_a = (a_\alpha, -b^\dagger_{\dot\alpha})$, $\bar{y}^a = (a^\dagger_\alpha, b_{\dot\alpha})$, $\theta_A = (c_r, d^\dagger_{\dot r})$, $\bar\theta^A = (c^\dagger_r, d_{\dot r})$, with indices $\alpha,\dot\alpha,r,\dot r$ taking values 1, 2 . In this notation, the basic representation, the singleton is denoted as $\mathcal{V}^{1,0}_{[0,1,0][0,0]}$. Its HWS $|Z\rangle$, i.e. the ground-state or ‘vacuum’, is chosen to be the scalar component $Z = \varphi^5 + i\varphi^6$ that satisfies

$$a_\alpha|Z\rangle = b_{\dot\alpha}|Z\rangle = c_r|Z\rangle = d_{\dot r}|Z\rangle = 0\,, \tag{4.1}$$

and is thus invariant under the non-semisimple superalgebra that combines $\mathfrak{iso}(4)_{ab}\times\mathfrak{iso}(4)_{cd}\times\mathfrak{u}(1)_{\Delta-J}\times\mathfrak{u}(1)_C$ with 24 supercharges (16 S’s and 8 Q’s). Clearly $|Z\rangle$ cannot be obtained from the $\mathfrak{su}(2,2|4)$ invariant trivial, but still physical, vacuum $|0\rangle$, associated to the identity operator, through the action of a finite number of oscillators.

Physical states in the singleton representation are given by all possible excitations $(a^\dagger)^{n_a}(b^\dagger)^{n_b}(c^\dagger)^{n_c}(d^\dagger)^{n_d}|Z\rangle$ satisfying the zero central charge condition

$$n_a - n_b + n_c - n_d = 0\,. \tag{4.2}$$

One can easily check that all elementary fields of $\mathcal{N}=4$ SYM and their derivatives can be represented in this way. The six scalars φ^i are given by the vacuum together with the excitations $c_r^\dagger d_{\dot r}^\dagger$, $c_1^\dagger c_2^\dagger d_1^\dagger d_2^\dagger$. The left-handed gaugini λ_A^α by the excitations $a_\alpha^\dagger d_{\dot r}^\dagger$ and $a_\alpha^\dagger c_r^\dagger d_1^\dagger d_2^\dagger$. The right-handed gaugini $\bar\lambda_{\dot\alpha}^A$ by $b_{\dot\alpha}^\dagger c_r^\dagger$ and $b_{\dot\alpha}^\dagger d_{\dot r}^\dagger c_1^\dagger c_2^\dagger$. The field strengths $F_{\mu\nu}^\pm$ by $a_\alpha^\dagger a_\beta^\dagger d_1^\dagger d_2^\dagger$, $b_{\dot\alpha}^\dagger b_{\dot\beta}^\dagger c_1^\dagger c_2^\dagger$. Finally, space-time derivatives are given by the action of $P_{\alpha\dot\alpha} = a_\alpha^\dagger b_{\dot\alpha}^\dagger$.

For the tensor product of L singletons, oscillators $a_\alpha^{(s)}$,$b_{\dot\alpha}^{(s)}$,$c_r^{(s)}$, $d_{\dot r}^{(s)}$ are to be thought as length L vectors with components acting at each of the L sites and trivial (anti-)commutation relations between oscillators acting on different sites. The vacuum Z^L is the tensor product of L copies of the singleton vacuum $|Z\rangle^L$. The Dynkin labels $[j,\bar\jmath][q_1,p,q_2]^{\Delta,B}$ of a length L SYM state made out of n_a, n_b, n_c and n_d oscillators follow from the relations

$$\begin{aligned}\Delta &= L+\frac{1}{2}n_a+\frac{1}{2}n_b\,, \quad B=\frac{1}{2}n_d-\frac{1}{2}n_c=|_{C=0}\,\frac{1}{2}n_a-\frac{1}{2}n_b\,,\\ [j,\bar\jmath\,] &= \left[\frac{1}{2}(n_{a_1}-n_{a_2}),\frac{1}{2}(n_{b_1}-n_{b_2})\right],\\ [q_1,p,q_2\,] &= [n_{c_2}-n_{c_1}, L-n_{c_2}-n_{d_1}, n_{d_1}-n_{d_2}]\,,\end{aligned} \tag{4.3}$$

with n_a, n_b, n_c, n_d, the total number of oscillators of a given type. In addition the zero central charge condition (4.2), i.e. $C_{(s)}=0$, is imposed at each site s.

4.1 Restricted semishort multiplets

The oscillator numbers n_a, n_b, n_c, n_d in (4.3) are required to be positive, since $|Z\rangle_L$ is annihilated by all raising operators. This simple condition imposes non-trivial bounds on the allowed $\mathfrak{psu}(2,2|4)$ charges in the SYM spectrum. For example $n_{a_2}+n_{c_1}\geq 0$ and $n_{b_2}+n_{d_2}\geq 0$ together with (4.2) imply the lower bounds

$$\Delta \;\geq\; 2j+\frac{3}{2}q_1+p+\frac{1}{2}q_2\,, \qquad \Delta \;\geq\; 2\bar\jmath+\frac{1}{2}q_1+p+\frac{3}{2}q_2\,, \tag{4.4}$$

for the conformal dimension of any state (not only HWS's!). In this section we will focus on states that simultaneously saturate the two bounds (4.4), or equivalently satisfy the intersection condition

$$\Delta = p+q_1+q_2+j+\bar\jmath\,. \tag{4.5}$$

This kind of states are only present in BPS and semishort multiplets. This can be seen by noting that the field content of any multiplet is generated by acting on the HWS with (a subset of) the 16 supercharges $\mathcal{Q}^A{}_\alpha$, $\bar{\mathcal{Q}}_{A\dot\alpha}$, cf. appendix A. The only supersymmetry charges among (A.3) whose weights violate the bounds (4.4) are $Q_1^+, \bar{Q}_4^+$ and they do so by exactly one unit. Therefore a state satisfying (4.5) should belong to a multiplet whose HWS has a conformal dimension that exceeds (4.5) by at most two units, i.e. $\Delta \leq 2+p+q_1+q_2+j+\bar{\jmath}$. This happens only for BPS or semishort multiplets. Indeed, the state under consideration could either be the HWS of a BPS multiplet that satisfies (4.5) and is annihilated by $Q_1^+, \bar{Q}_4^+$ or the level two superdescendant,

$$|\Psi_2\rangle = Q_1^+ \bar{Q}_4^+ |\Psi_0\rangle, \tag{4.6}$$

in a semishort multiplet whose HWS $|\Psi_0\rangle$ has $\Delta_0 = 2 + p + q_1 + q_2 + j + \bar{\jmath}$.

We will conveniently use states satisfying (4.5) as representatives of semishort and BPS multiplets. In terms of oscillators, this bound amounts to restricting attention to states for which

$$n_{a_2} = n_{b_2} = n_{c_1} = n_{d_2} = 0\,. \tag{4.7}$$

For simplicity, in the following, we denote the surviving oscillators (a_1, b_1, c_2, d_1) simply by (a, b, c, d). From (4.3) it follows that a SYM state with Dynkin labels $[j, \bar{\jmath}][q_1, p, q_2]^{\Delta,B}$ satisfying (4.5) carries

$$\Delta = L + j + \bar{\jmath}\,, \qquad L = p + q_1 + q_2 \qquad B = \frac{1}{2}(q_2 - q_1)\,, \tag{4.8}$$

and will be represented by the oscillator monomial

$$[j, \bar{\jmath}][q_1, p, q_2]^{\Delta,B} \equiv a^{2j}\, b^{2\bar{\jmath}}\, c^{q_1}\, d^{q_2}\, y^{p+q_1+q_2}\,. \tag{4.9}$$

The letters a, b, c, d here have a two-fold meaning. On the one hand they keep track of the quantum numbers $q_1, q_2, j, \bar{\jmath}$, on the other hand they describe how a given state is made out of oscillators a,b,c,d. Finally, the auxiliary variable y keeps track of p. Notice that for states satisfying (4.5), p is related to the number of letters $L = p + q_1 + q_2$ via (4.8), and therefore powers of y simultaneously count the number of letters, previously counted by u.

On these states, the residual superconformal symmetry is $\mathfrak{su}(1,1|2) \subset \mathfrak{psu}(2,2|4)$. The $\mathfrak{su}(1,1|2)$ raising operators among (2.18) are

$$Q_2^+ = \frac{a}{c}\,, \qquad \bar{Q}_2^+ = bc\,, \qquad Q_3^+ = ad\,, \qquad \bar{Q}_3^+ = \frac{b}{d}\,, \qquad P = ab\,, \qquad J = cd\,, \tag{4.10}$$

preserving the bound (4.5). Positive and negative powers in these expressions are associated to creation and annihilation operators respectively, e.g. $\frac{a}{c} \equiv a_1^\dagger c_2$, $\frac{b}{d} \equiv b_1^\dagger d_1$, and so on. It is then convenient to consider for BPS and semishort multiplets instead of the full character polynomials of $\mathfrak{psu}(2,2|4)$ and its bosonic subgroup $\mathfrak{so}(4) \times \mathfrak{so}(6)$, the restriction to states satisfying (4.5), giving rise to character polynomials of $\mathfrak{su}(1,1|2)$ and its bosonic subgroup $\mathfrak{sl}(2) \times \mathfrak{su}(2)$, respectively. We denote these as $\mathcal{V}^{\text{rst},\Delta}_{[j,\bar{\jmath}][q_1,p,q_2]}$ and $\chi^{\text{rst}}_{[j,\bar{\jmath}][k,p,q]}$, respectively. Discarding from now on $\mathfrak{sl}(2)$ descendants, i.e. total derivatives generated by $P = ab$, the character polynomial exclusively generated by the $\mathfrak{su}(2)$ raising operator $J = cd$ reads

$$\chi^{\text{rst}}_{[j,\bar{\jmath}][q_1,p,q_2]} = a^{2j} b^{2\bar{\jmath}} c^{q_1} d^{q_2} y^{p+q_1+q_2}\, \frac{1-(cd)^{p+1}}{1-cd}\,. \tag{4.11}$$

As discussed above, the restricted character polynomials $\mathcal{V}^{\text{rst},\Delta}_{[j,\bar{\jmath}][q_1,p,q_2]}$ is non-trivial only for BPS and semishort multiplets. For semishort multiplets one finds

$$\mathcal{V}^{\text{rst},\Delta}_{[j^*,\bar{\jmath}^*][q_1,p,q_2]} = \chi^{\text{rst}}_{[q_1+1,p,q_2+1](j+\frac{1}{2},\bar{\jmath}+\frac{1}{2})}\, T_{\text{short}} = y^2 abcd\, \chi^{\text{rst}}_{[q_1,p,q_2](j,\bar{\jmath})}\, T_{\text{short}}\,, \tag{4.12}$$

with

$$T_{\rm short} = (1-ad)(1-bc)\left(1-\frac{a}{c}\right)\left(1-\frac{b}{d}\right), \tag{4.13}$$

generated by the four $\mathfrak{psu}(1,1|2)$ supercharges (4.10). The factor y^2abcd takes care of the highest weight states of the restricted semishort multiplets, cf. (4.6), and maps $\mathfrak{psu}(1,1|2)$ primaries to semishort $\mathcal{N}=4$ superconformal primaries. The number of states inside the multiplet (4.12) is given by 2^4 times the restricted dimension of the highest weight state, i.e. $2^4\,(p+1)$. The nice factorized form (4.12) of the restricted semishort multiplet is to be contrasted with the more involved multiplicity formulae for semishort multiplets in $\mathfrak{psu}(2,2|4)$. We will make use of this restriction as a powerful simplifying tool in our analysis. The simplest generic multiplet of type (4.12) is the restriction of the short Konishi multiplet

$$\mathcal{V}^{\rm rst,2}_{[0,0][0,0,0]} = y^2abcd\,T_{\rm short}\,, \tag{4.14}$$

with total dimension 2^4. Notice that the state y^2abcd, corresponding to the weight $[101][\frac{1}{2},\frac{1}{2}]$, is the highest component of the Konishi current with $\Delta_0=3$ in the $\mathbf{15}=[1,0,1]$ of SU(4) that is a singlet ($p=0$) of $\mathfrak{su}(2)\subset\mathfrak{psu}(1,1|2)$.

The factorized formula (4.12) also holds for the $\frac{1}{4}$-BPS multiplets which are counted according to (A.8) below. In contrast, the restricted character polynomial corresponding to the $\frac{1}{2}$-BPS multiplet $\mathcal{V}^{\rm rst,\it n}_{[0^\dagger,0^\dagger][0n0]}$ is generated by J and the supersymmetry charges $Q^+_3,\bar{Q}^+_2$.[12] With (4.10) one finds:

$$\begin{aligned}\mathcal{V}^{{\rm rst},n}_{[0^\dagger,0^\dagger][0,n,0]} &= \chi^{\rm rst}_{[0,0][0,n,0]} - \chi^{\rm rst}_{[\frac{1}{2},0][0,n,1]} - \chi^{\rm rst}_{[0,\frac{1}{2}][1,n,0]} + \chi^{\rm rst}_{[\frac{1}{2},\frac{1}{2}][1,n,1]}\\ &= y^n\,\frac{(1-ad)(1-bc)-(cd)^n(a-c)(b-d)}{(1-cd)}\,.\end{aligned} \tag{4.15}$$

4.2 The semishort primary sieve

Here we derive multiplicity formulae for semishort-semishort $\mathfrak{psu}(2,2|4)$ multiplets in $\mathcal{N}=4$ SYM theory. According to (4.7) the spectrum of single-letter SYM words saturating the bound (4.5) consists of all possible excitations satisfying (4.7). The multiplicities of these states can be derived via Polya theory. The basic ingredient is the one-letter partition function:

$$\mathcal{Z}^{\rm rst}_1 = y\,\frac{1+cd-ad-bc}{1-ab}\,, \tag{4.16}$$

obtained from (3.1) upon restriction. The four terms in the numerators corresponds to the elementary SYM fields saturating the bound (two scalars and two fermionic components) while the expansion of the denominator generates their derivatives. The restricted partition function is given by Polya's formula (3.3):

$$\mathcal{Z}^{\rm rst}_n = y^n(1-ab)\sum_{d|n}\frac{\varphi(d)}{n}\left[\frac{1+(cd)^d-(ad)^d-(bc)^d}{1-(ab)^d}\right]^{n/d}\,, \tag{4.17}$$

[12] The full $\frac{1}{2}$-BPS multiplet is generated by $Q^\pm_{3,4},\bar{Q}^\pm_{1,2}$ supersymmetries and $\mathfrak{su}(4)\times\mathfrak{so}(4)$ charges, cf. appendix A.

The factor $(1-ab)$ removes total derivatives, in much the same way as in (3.4). The restricted polynomial (4.17) contains only contributions coming from $\frac{1}{2}$-BPS and semishort multiplets. This can be checked by noticing that once the BPS series $\sum_n \mathcal{V}^{\mathrm{rst},n}_{[0^\dagger,0^\dagger][0,n,0]}$ is subtracted, the spectrum organizes into multiplets of the type (4.12). Specifically, the difference $(\mathcal{Z}^{\mathrm{rst}}_n - \mathcal{V}^{\mathrm{rst},n}_{[0^\dagger,0^\dagger][0,n,0]})$ vanishes at the four zeros of (4.13)

$$\left(\mathcal{Z}^{\mathrm{rst}}_n - \mathcal{V}^{\mathrm{rst},n}_{[0^\dagger,0^\dagger][0,n,0]}\right)\Big|_{a=c,\frac{1}{d}} = (\mathcal{Z}^{\mathrm{rst}}_n - \mathcal{V}^{\mathrm{rst},n}_{[0^\dagger,0^\dagger][0,n,0]})\Big|_{b=d,\frac{1}{c}} = 0\,, \tag{4.18}$$

as follows from the remarkable identity $\sum_{n|d}\varphi(d) = n$. Semishort primaries can then be isolated by factoring out T_{short}. More precisely,

$$\mathcal{Z}^{\mathrm{short}}_{n,\mathrm{suprim}} \equiv (y^2\,abcd\,T_{\mathrm{short}})^{-1}\left(\mathcal{Z}^{\mathrm{rst}}_n - \mathcal{V}^{\mathrm{rst}}_{[0^\dagger,0^\dagger][0,n,0]}\right)_{\mathrm{HW}} + \frac{y^{n-2}}{a^2b^2}\,, \tag{4.19}$$

is a regular rational function describing the character polynomial of superprimaries sitting in semishort and BPS multiplets in the n-letter spectrum of SYM states. $(y^2\,abcd\,T_{\mathrm{short}})^{-1}$ disposes of supersymmetry descendants according to (4.12).[13] The subscript HW denotes the reduction to $\mathfrak{su}(2)$ highest weight states given by dividing out the $\mathfrak{su}(2)$ multiplets (4.11). This can be done by counting states according to the rule

$$y^{p+q_1+q_2}c^{q_1}d^{q_2} \to \begin{cases} y^{p+q_1+q_2}c^{q_1}d^{q_2} & p\geq 0 \\ -y^{p+q_1+q_2}c^{q_1+p+1}d^{q_2+p+1} & p<0 \end{cases}\,, \tag{4.20}$$

isolating $\mathfrak{su}(2)$ HWS's. Alternatively the same result is found by multiplying $\mathcal{Z}^{\mathrm{rst}}_{n,\mathrm{suprim}}$ by $(1-cd)$ and then deleting all bosons (fermions) coming with negative (positive) multiplicities. The term y^{n-2}/a^2b^2 in (4.19) accounts for $\frac{1}{2}$-BPS primaries with weights $[-1,-1][0,n-2,0] = [00][0n0]$ according to (A.9). Notice that here powers of y are no longer related to the number of letters (powers of ℓ) since semishort primaries do not belong to the $\mathfrak{psu}(1,1|2)$ sector.

For the lowest values of L, the above procedure yields

$$\begin{aligned}
\mathcal{Z}^{\mathrm{short}}_{\square\square,\mathrm{suprim}} &= \frac{1}{a^2b^2(1-a^2b^2)}\,,\\
\mathcal{Z}^{\mathrm{short}}_{\square\square\square,\mathrm{suprim}} &= \frac{y\,(\frac{1}{a^2b^2}-\frac{c}{a}-\frac{d}{b})}{(1-a^2b^2)\,(1-a^3b^3)}\,,\\
\mathcal{Z}^{\mathrm{short}}_{\substack{\square\\\square},\mathrm{suprim}} &= \frac{y\,a^3b^3\,(\frac{1}{a^2b^2}-\frac{c}{a}-\frac{d}{b})}{(1-a^2b^2)\,(1-a^3b^3)}\,,\\
\mathcal{Z}^{\mathrm{short}}_{\square\square\square\square,\mathrm{suprim}} &= \frac{y^2\,(1+cd\,(a^3b^3+a^5b^5+a^8b^8)+c^2a^7b^9+d^2a^9b^7)}{a^2b^2\,(1-a^2b^2)\,(1-a^3b^3)\,(1-a^4b^4)} - \\
&\quad - \frac{y^2\,(cb+da)(a^2b^2+a^3b^3+a^4b^4-a^6b^6)}{(1-a^2b^2)\,(1-a^3b^3)\,(1-a^4b^4)}\,,
\end{aligned}$$

[13]In particular the factor $y^2\,abcd$ map $\mathfrak{su}(1,1|2)$ HWS to superconformal primaries via (4.6).

$$\mathcal{Z}^{\text{short}}_{\boxplus,\text{suprim}} = \frac{y^2\,(1 + cd\,(\frac{1}{ab} + a^2b^2 + a^3b^3) + c^2ab^3 + d^2a^3b - (cb + da)\,(1 + ab))}{(1 - a^2b^2)^2(1 - a^3b^3)}\,,$$

$$\mathcal{Z}^{\text{short}}_{\boxplus,\text{suprim}} = \frac{y^2(ab + cd\,(1 + ab + a^2b^2) + c^2b^2 + d^2a^2 - (cb + da)\,(\frac{1}{ab} + a^2b^2))}{(1 - ab)(1 - a^2b^2)(1 - a^4b^4)}\,. \quad (4.21)$$

Continuing to higher L, the complete list of semishort multiplets appearing in the $\mathcal{N} = 4$ SYM spectrum is obtained. The $\mathfrak{su}(2)^2 \times \mathfrak{su}(4)$ charges can be read off from (4.9) i.e. $[\frac{n_a}{2}, \frac{n_b}{2}][n_c, n_y - n_c - n_d, n_d]$, while

$$\Delta = 2 + n_y + \frac{1}{2}(n_a + n_b)\,, \qquad B = \frac{1}{2}(n_d - n_c)\,, \tag{4.22}$$

and L is specified by the subscript of $\mathcal{Z}$'s. The results for $L = 2, 3$ precisely match (3.8) with the coefficients c_n given by the expansion

$$\sum_{n=0}^{\infty} c_n\, x^n = \frac{1}{(1 - x^2)(1 - x^3)}\,. \tag{4.23}$$

Notice that representatives for a given YT-pleton can be always chosen inside the $\mathfrak{hs}(1,1)$ subgroup. This corresponds to setting $c = d = 0$ in (4.21).

Similar techniques can be applied to the study of any closed subsector in the SYM spectrum. For example SU(4) singlets in the AC series are described by states saturating one of the bounds (4.4) and $q_1 = p = q_2 = 0$. The various conditions for the first bound combine to give

$$n_{c_1} = n_{c_2} = n_{a_2} = 0 \qquad n_{d_1} = n_{d_2} = L\,. \tag{4.24}$$

This leads to the $\mathfrak{su}(2,1)$ invariant subsector with letters $a_1^{\dagger\,2+n}\, b_1^{\dagger\,m}\, b_2^{\dagger\,n-m}\, d_1^\dagger d_2^\dagger |Z\rangle$, which are essentially derivatives of the self-dual field strength $D_{11}^m D_{12}^{n-m} F_{11}$. Anomalous dimensions for three-letter states of this type will be computed in the next section using the corresponding $\mathfrak{su}(1,2)$ spin chain.

4.2.1 Semishort multiplets group into long multiplets

We can now explicitly show that the semi-short multiplets appearing in the free $\mathcal{N} = 4$ SYM spectrum above organize into long multiplets. This is expected since after switching on interactions the shortening conditions (A.5) are generically no longer satisfied. Specifically, a semishort multiplet appearing in the decomposition of an L-pleton joins two multiplets from the $(L+1)$-pleton and a fourth one from the $(L+2)$-pleton to build a long multiplet according to (A.7). The semishort multiplets appearing in this decomposition are related to each other by the action of Q^1_- and $\bar{Q}_{4-}$ in (A.3).

Our statement then is equivalent to claiming that the total partition function of the semi-short SYM spectrum

$$\mathcal{Z}^{\text{rst}} = \sum_{n=2}^{\infty} \mathcal{Z}^{\text{rst}}_n\,, \tag{4.25}$$

after subtraction of the $\frac{1}{2}$-BPS multiplets contains the factors $(1 - y\frac{c}{a})$ and $(1 - y\,\frac{d}{b})$. To prove this, we write the total partition function as

$$\begin{aligned}\mathcal{Z}^{\rm rst} &= -(1-ab)\sum_{k=1}^{\infty}\frac{\varphi(k)}{k}\ln\left\{1 - y^k\left(\frac{1+(cd)^k-(ad)^k-(bc)^k}{1-(ab)^k}\right) - \right\} - \\ &\quad [.5ex] - y\,(1+cd-ad-bc)\,,\end{aligned} \tag{4.26}$$

while for the total partition function of $\frac{1}{2}$-BPS multiplets we obtain

$$\mathcal{Z}^{\rm rst}_{\frac{1}{2}-{\rm BPS}} = \sum_{n=2}^{\infty}\mathcal{V}^{\rm rst,n}_{[0^\dagger,0^\dagger][0,n,0]} = \frac{y^2(1-ad)(1-bc)}{(1-y)(1-cd)} - \frac{(cdy)^2\,(a-c)(b-d)}{(1-cd)(1-ycd)}\,. \tag{4.27}$$

Using

$$\sum_{k=1}^{\infty}\frac{\varphi(k)}{k}\ln(1-x^k) = -\frac{x}{1-x}\,, \tag{4.28}$$

one finds indeed that

$$\left(\mathcal{Z}^{\rm rst}|_{y\to\frac{a}{c}} - \mathcal{Z}^{\rm rst}_{\frac{1}{2}-{\rm BPS}}\right)\Big|_{y\to\frac{a}{c}} = 0\,, \tag{4.29}$$

and likewise for $y \to \frac{b}{d}$. Hence, the semishort multiplets in the free SYM spectrum organize in long multiplets whose highest weight states are collected in the regular function

$$\mathcal{Z}^{\rm long}_{\rm suprim} \equiv \frac{1}{y^2\,abcd}\,T^{-1}_{\rm long}\left(\mathcal{Z}^{\rm rst} - \mathcal{Z}^{\rm rst}_{\frac{1}{2}{\rm BPS}}\right)_{\rm HW} + \frac{1}{a^2b^2}\,\frac{1}{1-y}\,, \tag{4.30}$$

with

$$T_{\rm long} = \left(1 - y\frac{c}{a}\right)\left(1 - y\frac{d}{b}\right)(1-ad)(1-bc)\left(1-\frac{a}{c}\right)\left(1-\frac{b}{d}\right), \tag{4.31}$$

defining the restriction of the long Konishi multiplet.

5. Symmetry breaking and anomalous dimensions

In the interacting theory only one out of the infinite tower of conserved current doubleton multiplets

$$\mathcal{Z}_{\square\!\square} = \sum_{n=0}^{\infty}\mathcal{V}_{2n}\,, \qquad \mathcal{V}_j := \mathcal{V}^{j,0}_{[-1+\frac{1}{2}j^*,-1+\frac{1}{2}j^*][0,0,0]}\,, \tag{5.1}$$

is protected against quantum corrections to the scaling dimension: the $\mathcal{N} = 4$ supercurrent multiplet $\mathcal{V}_0 = \mathcal{V}^{2,0}_{[0^\dagger,0^\dagger][0^\dagger,2,0^\dagger]}$. The remaining multiplets $\mathcal{V}_{2n}$ acquire anomalous dimensions which violate the conservation of their HS currents at the quantum level. At one-loop, one has [96, 97, 40]

$$\gamma_{1-{\rm loop}}(2n) = \frac{g^2_{\rm YM}N}{2\pi^2}\,h(2n), \qquad h(j) = \sum_{k=1}^{j}\frac{1}{k}\,. \tag{5.2}$$

This elegant ('number theoretic') formula gives a clue on how to compute generic anomalous dimensions at first order in perturbation theory relying on symmetry breaking considerations. Naively, one would look for all occurrences of the broken currents $\mathcal{V}_{2n}$ within some

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operator $\mathcal{O}$. Each occurrence of some broken current should contribute to the anomalous dimension of $\mathcal{O}$ a term proportional to $h(2n)$. Indeed, this is nearly what happens, the one-loop dilatation operator [74] can be written as

$$H = \sum_{s=1}^{L} H_{(s,s+1)} = \sum_{s=1}^{L} \sum_{j=0}^{\infty} 2h(j)\, P^{j}_{(s,s+1)}\,, \tag{5.3}$$

where $P^{j}_{(s,s+1)}$ projects the product of fields ('letters') at nearest neighboring sites s and $s+1$ onto $\mathcal{V}_j$. Here, the sum goes over all values of j and not just the even ones. The point is that although bilinear currents $\mathcal{V}_{2n+1}$ corresponding to the broken generators are eliminated after tracing over color indices, they still appear in subdiagrams $\boxminus$ inside a bigger trace. The corresponding decomposition for doubletons is given in (3.8).

5.1 'Twist' three anomalous dimensions

Expressions (4.21) give not only multiplicities and charges of semishort primaries but also a representative of each multiplet in terms of the oscillators (a_1, b_1, c_2, d_1). For instance states in the $\mathfrak{sl}(2)$ sector inside $\mathfrak{psu}(1,1|2)$, associated to words made out of powers of $a_1 b_1$ (i.e. a single scalar and all its derivatives) can be taken as representatives for semishort multiplets $\mathcal{V}_{0,n}$. The letters in this sector are:

$$|k\rangle_0 = \left(a_1^\dagger b_1^\dagger\right)^k |Z\rangle \quad \leftrightarrow \quad \mathcal{D}_1^k Z\,, \tag{5.4}$$

with $\mathcal{D}_i = D_{\alpha=i,\dot\alpha=i}$. Similarly derivatives of $(a^\dagger d^\dagger)^3$ appearing in $Q_1^+ \bar{Q}_4^+ T_{\rm short}\, \mathcal{Z}^{\rm rst}_{3,\rm suprim}$ can be chosen as representatives of $\mathcal{V}_{1,n}$ (similarly for the conjugate multiplets $\mathcal{V}_{-1,n}$). Indeed there is a single state of this type inside each fermionic semishort multiplet in $\mathcal{Z}_3^{\rm rst}$. The letters are now:

$$|k\rangle_1 = \left(a_1^\dagger b_1^\dagger\right)^k \left(a_1^\dagger d_1^\dagger\right) |Z\rangle \quad \leftrightarrow \quad \mathcal{D}_1^k \lambda\,, \tag{5.5}$$

with $\lambda = \lambda_{\alpha=1,\dot r=1}$ one of the gaugini. The $\mathfrak{sl}(2)$ generators in both cases can be written as

$$J_- = a_1 b_1\,, \qquad J_+ = a_1^\dagger b_1^\dagger\,, \qquad J_3 = \frac{1}{2} a_1 a_1^\dagger - \frac{1}{2} b_1^\dagger b_1\,, \tag{5.6}$$

while the spin is given by $J_3 |k\rangle_m = (\frac{1}{2} + \frac{m}{2})|k\rangle_m$. Therefore for $\mathcal{V}_{m,n}$, $m = 0, 1$, we use the $\mathfrak{sl}(2)$ spin chain with spin $\frac{1}{2} + \frac{1}{2}m$. We use a unified notation for a single spin state of either chains

$$|k\rangle_m = (a_1^\dagger)^{k+m} (b_1^\dagger)^k (d_1^\dagger)^m |Z\rangle \quad \leftrightarrow \quad m\mathcal{D}_1^k \lambda + (1-m)\mathcal{D}_1^k Z\,. \tag{5.7}$$

The hamiltonian of the relevant (super) spin chains in the two subsectors can be computed using the harmonic action in [74]. The resulting hamiltonian 'density' is

$$H_{(12)} |k, n-k\rangle_m = \sum_{k'=0}^{n} c^{(m)}_{n,k,k'} |k', n-k'\rangle_m\,, \tag{5.8}$$

$n\backslash m$	0	1	2	3	4	5	6
3	$\frac{15}{16}$	$\frac{5}{4}$	$\frac{47}{32}$	$\frac{131}{80}$	$\frac{71}{40}$	$\frac{1059}{560}$	$\frac{4461}{2240}$
5	$\frac{35}{32}$	$\frac{133}{96}$	$\frac{761}{480}$	$\frac{487}{280}$	$\frac{12533}{6720}$	$\frac{39749}{20160}$	$\frac{13873}{6720}$
6	$\frac{227}{160}$	$\frac{761}{480}$	$\frac{967}{560}$	$\frac{2069}{1120}$	$\frac{39349}{20160}$	$\frac{2747}{1344}$	$\frac{3929}{1848}$
7	$\frac{581}{480}$	$\frac{179}{120}$	$\frac{3763}{2240}$	$\frac{18383}{10080}$	$\frac{39133}{20160}$	$\frac{7543}{3696}$	$\frac{94373}{44352}$
8	$\frac{5087}{3360}$	$\frac{1403}{840}$	$\frac{18187}{10080}$	$\frac{38677}{20160}$	$\frac{49711}{24640}$	$\frac{2593}{1232}$	$\frac{629227}{288288}$

Table 2: First few paired anomalous dimensions for $\mathcal{V}_{m,n}$ with $L = 3$.

with coefficients

$$c^{(m)}_{n,k,k'} = \begin{cases} h(k+m) + h(n-k+m) & \text{for } k = k' \\ \dfrac{k!(n-k+m)!}{k'!(n-k'+m)!(k-k')} & \text{for } k > k', \\ \dfrac{(n-k)!(k+m)!}{(n-k')!(k'+m)!(k'-k)} & \text{for } k < k'. \end{cases} \tag{5.9}$$

For $m = 0$ this is equivalent to the $\mathfrak{sl}(2)$ subsector of letters $\mathcal{D}^n Z$ up to a rescaling by $n!$

For multiplets $\mathcal{V}_{m,n}$ with $m \geq 2$ we use the $\mathfrak{su}(1,2)$ spin chain corresponding to the closed subsector with residual symmetry algebra $\mathfrak{su}(1,2)$. The spin states are now specified by two conserved charges corresponding to the rank two algebra $\mathfrak{su}(1,2)$

$$|k,l\rangle = (a_1^\dagger)^{2+k+l}(b_1^\dagger)^k(b_2^\dagger)^l d_1^\dagger d_2^\dagger |Z\rangle \quad \leftrightarrow \quad \mathcal{D}_1^k \mathcal{D}_2^l \mathcal{F}\,, \tag{5.10}$$

with $\mathcal{F} = F_{\alpha=1,\beta=1}$. The planar, one-loop dilatation generator H acts on two adjacent spin sites as

$$H_{(12)}|k,l;m-k,n-l\rangle = \sum_{r=0}^{m}\sum_{s=0}^{n} c^{m,n}_{k,l;r,s}|r,s;m-r,n-s\rangle\,, \tag{5.11}$$

with

$$c^{m,n}_{k,l;r,s} = \begin{cases} h(2+k+l) + h(2+n+m-k-l), & \text{for } k = r, l = s\,, \\ -\dfrac{k!l!(2+n+m-k-l)!(k+l-r-s-1)!}{r!s!(2+n+m-r-s)!(k-r)!(l-s)!}, & \text{for } k \geq r, l \geq s\,, \\ -\dfrac{(m-k)!(n-l)!(2+k+l)!(r+s-k-l-1)!}{(m-r)!(n-s)!(2+r+s)!(r-k)!(s-l)!}, & \text{for } k \leq r, l \leq s, \\ 0, & \text{for } k > r, l < s\,, \\ 0, & \text{for } k < r, l > s\,. \end{cases} \tag{5.12}$$

The coefficients again follow from the harmonic action.

We now compute the spectrum of one-loop planar anomalous dimensions explicitly using (5.8)–(5.12).[14] By inspecting the spectrum of lowest-lying states and their energies we find that almost all of them form pairs with degenerate energies. We list the pairs

$n\backslash m$	0	1	2	3	4	5	6
0	0	$\frac{3}{4}$	$\frac{9}{8}$	$\frac{11}{8}$	$\frac{25}{16}$	$\frac{137}{80}$	$\frac{147}{80}$
2	$\frac{1}{2}$	$\frac{25}{24}$	$\frac{4}{3}$	$\frac{123}{80}$	$\frac{407}{240}$	$\frac{3067}{1680}$	$\frac{271}{140}$
4	$\frac{3}{4}$	$\frac{49}{40}$	$\frac{71}{48}$	$\frac{929}{560}$	$\frac{9}{5}$	$\frac{9661}{5040}$	$\frac{2259}{1120}$
6	$\frac{11}{12}$	$\frac{761}{560}$	$\frac{191}{120}$	$\frac{8851}{5040}$	$\frac{66}{35}$	$\frac{221047}{110880}$	$\frac{21031}{10080}$
8	$\frac{25}{24}$	$\frac{7381}{5040}$	$\frac{101}{60}$	$\frac{101861}{55440}$	$\frac{6581}{3360}$	$\frac{329899}{160160}$	$\frac{21643}{10080}$
10	$\frac{137}{120}$	$\frac{86021}{55440}$	$\frac{493}{280}$	$\frac{2748871}{1441440}$	$\frac{20383}{10080}$	$\frac{203545}{96096}$	$\frac{122029}{55440}$

Table 3: First few unpaired anomalous dimensions for $\mathcal{V}_{m,n}$ with $L = 3$. Parity is given by $(-1)^{m+1}$.

in table 2.[15] For the unpaired states one can observe a pattern in the table of energies, table 3. We find that all energies agree with the formula

$$\delta D = \frac{g_{\rm YM}^2 N}{8\pi^2}\left(2h\left(\frac{1}{2}m - \frac{1}{2}\right) + 2h\left(m + \frac{1}{2}n\right) + 2h\left(\frac{1}{2}m + \frac{1}{2}n\right) - 2h\left(-\frac{1}{2}\right)\right). \tag{5.13}$$

In particular, for $m = 1$ the energies are

$$\delta D = \frac{g_{\rm YM}^2 N}{8\pi^2}\left(+2h\left(1 + \frac{1}{2}n\right) + 2h\left(\frac{1}{2} + \frac{1}{2}n\right) - 2h\left(-\frac{1}{2}\right)\right) = \frac{g_{\rm YM}^2 N}{2\pi^2}\, h(n+2)\,, \tag{5.14}$$

which agrees precisely with the energy of the short twist 2 multiplet $\mathcal{V}_{2n+2}$, (5.1). Superconformal invariance requires this degeneracy so that the short multiplets can join to form a long multiplet. The cases $m = 0$ and $n = 0$ also seem interesting, we find $\delta D = (g_{\rm YM}^2 N/8\pi^2) 4h(\frac{1}{2}n)$ and $\delta D = (g_{\rm YM}^2 N/8\pi^2) 6h(m)$.

Let us note a peculiarity of the three parton states discussed above. Intriguingly, for $\mathcal{V}_{m,n}$ we can reproduce all $\mathfrak{su}(2,1)$ spin chain results also with a $\mathfrak{sl}(2)$ spin chain with spin $-m/2 - 1/2$ and n excitations given by (5.8), (5.9).

6. Conclusions

In the present paper, we have studied the decomposition of the spectrum of single-trace gauge invariant operators of free $\mathcal{N} = 4$ SYM theory with $\mathrm{SU}(N)$ gauge group in irreps of $\mathfrak{hs}(2,2|4)$, the HS extension of the superconformal algebra $\mathfrak{psu}(2,2|4)$. To this end we have shown that HS L-pleton multiplets can be associated to Young tableaux made of L boxes, each representing a singleton of $\mathfrak{psu}(2,2|4)/\mathfrak{hs}(2,2|4)$, compatible with the cyclicity of the trace over color indices. For other gauge groups, further restrictions are to be imposed. For $L = 2$ only the symmetric product gives rise to physical operators independently of the choice of the (simple) gauge group [34]. The antisymmetric doubleton is ruled out by the cyclicity of the trace but still its decomposition is relevant to diagrammatic computations of composite operators where such combinations appear in intermediate channels. We

[14]The hamiltonian is related to the dilatation operator by $\delta D = (g_{\rm YM}^2 N/8\pi^2) H + \mathcal{O}(g_{\rm YM}^3)$.

[15]The energies are all rational numbers because there is always just a single pair up to $n \le 8$. Starting from $n = 9$ there is more than one pair and the energies become irrational.

have then focussed on tripletons associated to Young tableaux with $L = 3$ boxes. The only tableaux compatible with the cyclicity of the trace are the totally symmetric (d_{abc}) and antisymmetric (f_{abc}) tripletons. The former includes the KK recurrences of the doubleton and the latter part of the Goldstone fields. The remaining Goldstone fields belong (in the free theory) to the $L = 4$-letter 'window' $\boxplus$.

For higher L-pletons we have identified all operators belonging to BPS or semishort-semishort multiplets of $\mathfrak{psu}(2,2|4)$ in the free theory. In particular, we have derived the partition function for $\mathcal{N} = 4$ superconformal primaries saturating both left and right unitarity bounds. After interactions are turned on, they are shown to combine such as to give rise to long multiplets of the superconformal group as expected from the boundary description of the 'Grande Bouffe' in the AdS bulk.

Finally, we have computed anomalous dimensions of operators that appear in the decomposition of tripletons in terms of $\mathfrak{psu}(2,2|4)$ multiples. Remarkably the resulting anomalous dimensions for the full tripleton tower follow from integrable spin chains with symmetry group $\mathfrak{sl}(2)_j$ and arbitrarily high spin j. The regularity of the pattern suggests the presence of some not-so 'hidden' symmetry. Indeed there are by now various independent indications that some aspects of the dynamics of large N $\mathcal{N} = 4$ SYM theory and its holographic dual type IIB superstring on $AdS_5 \times S^5$ expose an integrable structure. In the latter, the supercoset structure of the target superspace and the (generalized) flatness of the supercoset currents allow one to identify an infinite number of conserved charges that form a yangian [98]–[100]. In the former, the dilatation operator can be identified with the hamiltonian of an integrable super-spin chain to lowest orders. Some of the infinite charges have been identified and given explicit perturbative expressions. These two routes to integrability have been connected in [101]. In the emergence of the integrable structure HS symmetry enhancement has so far played only a marginal role. Yet HS dynamics in lower dimensions is typically formulated in terms of a Cartan integrable system [23, 83, 84, 34, 35, 15]. It is then tempting to speculate that at least at one loop and large N, HS symmetry could explain the pattern of mass-shifts and anomalous dimensions and give some additional insight into the geometric origin of integrability.

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A. $\mathcal{N} = 4$ shortening, yet again

Here we collect some notation for representations of the $\mathcal{N} = 4$ superconformal algebra $\mathfrak{psu}(2,2|4)$ and their shortenings. We denote by

$$\mathcal{V}^{\Delta,B}_{[j,\bar{j}][q_1,p,q_2]}\,, \tag{A.1}$$

a generic long multiplet of $\mathfrak{psu}(2,2|4)$ with highest weight state in the $\mathcal{R}_{[j,\bar{\jmath}][q_1,p,q_2]}$ representation of $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{su}(4)$, conformal dimension Δ and hypercharge B. As above, $[q_1,p,q_2]$ are Dynkin labels of $\mathfrak{su}(4)$ while $[j,\bar{\jmath}]$ denote the spins under $\mathfrak{su}(2)\times\mathfrak{su}(2)$.

The representation content of the long multiplet (A.1) under the bosonic subalgebra $\mathfrak{su}(2)\times\mathfrak{su}(2)\times\mathfrak{su}(4)$ may be found from evaluating the tensor product $\mathcal{V}^{2,0,0}_{[0,0][0,0,0]}\chi^{\Delta-2,B,P}_{[j,\bar{\jmath}][q_1,p,q_2]}$, with the long Konishi multiplet $\mathcal{V}^{2,0,0}_{[0,0][0,0,0]}$, or explicitly by using the Racah-Speiser algorithm as

$$\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} = \sum_{\epsilon_{A\alpha},\bar{\epsilon}^A_{\dot{\alpha}}\in\{0,1\}} \chi_{[j,\bar{\jmath}][q_1,p,q_2]+\epsilon_{A\alpha}\mathcal{Q}^A_\alpha+\bar{\epsilon}^A_{\dot{\alpha}}\bar{\mathcal{Q}}_{A\dot{\alpha}}}\,, \tag{A.2}$$

with the sum running over the 2^{16} combinations of the 16 supersymmetry charges $\mathcal{Q}^A{}_\alpha$, $\bar{\mathcal{Q}}_{A\dot{\alpha}}$, $A=1,\ldots,4,\alpha,\dot{\alpha}=1,2$ with

Dynkin labels[16]

$$\begin{aligned}
\mathcal{Q}^1{}_\alpha &= a^\dagger_\alpha\,c_1 \equiv \frac{a_\alpha}{c_1} = \left[\pm\frac{1}{2},0\right][1,0,0]\,, &\quad \bar{\mathcal{Q}}_{1\dot{\alpha}} &= b^\dagger_{\dot{\alpha}}\,c^\dagger_1 \equiv b_{\dot{\alpha}}\,c_1 = \left[0,\pm\frac{1}{2}\right][-1,0,0]\\
\mathcal{Q}^2{}_\alpha &= a^\dagger_\alpha\,c_2 \equiv \frac{a_\alpha}{c_2} = \left[\pm\frac{1}{2},0\right][-1,1,0]\,, &\quad \bar{\mathcal{Q}}_{2\dot{\alpha}} &= b^\dagger_{\dot{\alpha}}\,c^\dagger_2 \equiv b_{\dot{\alpha}}\,c_2 = \left[0,\pm\frac{1}{2}\right][1,-1,0]\,,\\
\mathcal{Q}^3{}_\alpha &= a^\dagger_\alpha\,d^\dagger_1 \equiv a_\alpha\,d_1 = \left[\pm\frac{1}{2},0\right][0,-1,1]\,, &\quad \bar{\mathcal{Q}}_{3\dot{\alpha}} &= b^\dagger_{\dot{\alpha}}\,d_1 \equiv \frac{b_{\dot{\alpha}}}{d_1} = \left[0,\pm\frac{1}{2}\right][0,1,-1]\,,\\
\mathcal{Q}^4{}_\alpha &= a^\dagger_\alpha\,d^\dagger_2 \equiv a_\alpha\,d_2 = \left[\pm\frac{1}{2},0\right][0,0,-1]\,, &\quad \bar{\mathcal{Q}}_{4\dot{\alpha}} &= b^\dagger_{\dot{\alpha}}\,d_2 \equiv \frac{b_{\dot{\alpha}}}{d_2} = \left[0,\pm\frac{1}{2}\right][0,0,1]\,.
\end{aligned} \tag{A.3}$$

Every $\mathcal{Q}^A{}_\alpha$, $\bar{\mathcal{Q}}_{A\dot{\alpha}}$ raises the conformal dimension by $\frac{1}{2}$, parity is left invariant, and the hypercharge B is lowered and raised by $\frac{1}{2}$ by each $\mathcal{Q}^A{}_\alpha$ and $\bar{\mathcal{Q}}_{A\dot{\alpha}}$ respectively. In order to make sense out of (A.2) also for small values of $q_1,p,q_2,j,\bar{\jmath}$, we note that the character polynomials associated with negative Dynkin labels are defined according to

$$\begin{aligned}
\chi_{[j,\bar{\jmath}][q_1,p,q_2]} &= -\chi_{[j,\bar{\jmath}][-q_1-2,p+q_1+1,q_2]} = -\chi_{[j,\bar{\jmath}][q_1,p+q_2+1,-q_2-2]}\\
&= -\chi_{[j,\bar{\jmath}][q_1+p+1,-p-2,q_2+p+1]}\\
&= -\chi_{[-j-1,\bar{\jmath}][q_1,p,q_2]} = -\chi_{[j,-\bar{\jmath}-1][q_1,p,q_2]}\,.
\end{aligned} \tag{A.4}$$

In particular, this implies that $\chi_{[j,\bar{\jmath}][q_1,p,q_2]}$ is identically zero whenever any of the weights q_1, p, q_2 takes the value -1 or one of the spins j, $\bar{\jmath}$ equals $-\frac{1}{2}$.

In $\mathcal{N}=4$ SYM, there are two types of (chiral) shortening conditions for particular values of the conformal dimension Δ: BPS (B) which may occur when at least one of the spins is zero, and semi-short (C) ones. The corresponding multiplets are constructed similar to the long ones (A.2), with the sum running only over a restricted number of supersymmetries. Specifically, the critical values of the conformal dimensions and the

[16]Notice the flip of notations for the conjugate charges with respect to [29] and the unconventional use of oscillators in the denominator to mean annihilation operators.

restrictions on the sums in (A.2) are given by

$$\begin{array}{llll}
\mathrm{B}_L: & \mathcal{V}^{\Delta,B}_{[0^\dagger,\bar{\jmath}][q_1,p,q_2]} & \Delta = p + \frac{3}{2}q_1 + \frac{1}{2}q_2 & \epsilon_{1\pm} = 0 \\
\mathrm{B}_R: & \mathcal{V}^{\Delta,B}_{[j,0^\dagger][q_1,p,q_2]} & \Delta = p + \frac{1}{2}q_1 + \frac{3}{2}q_2 & \bar{\epsilon}_{4\pm} = 0 \\
\mathrm{C}_L: & \mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][q_1,p,q_2]} & \Delta = 2 + 2j + p + \frac{3}{2}q_1 + \frac{1}{2}q_2 & \epsilon_{1-} = 0 \\
\mathrm{C}_R: & \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,p,q_2]} & \Delta = 2 + 2\bar{\jmath} + p + \frac{1}{2}q_1 + \frac{3}{2}q_2 & \bar{\epsilon}_{4-} = 0
\end{array} \tag{A.5}$$

for the different types of multiplets. They represent the basic $\frac{1}{8}$-BPS and $\frac{1}{16}$ semishortenings in $\mathcal{N}=4$ SCA and are indicated as in with a "$\dagger$" and a "$*$" respectively.

If the conformal dimension Δ of the HWS of a long multiplet (A.1) satisfies one of the conditions (A.5), the multiplet splits according to

$$\begin{aligned}
\mathrm{L}: \quad & \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} = \mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][q_1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B-\frac{1}{2}}_{[j-\frac{1}{2}^*,\bar{\jmath}][q_1+1,p,q_2]}\,, \\
\mathrm{R}: \quad & \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} = \mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B+\frac{1}{2}}_{[j,\bar{\jmath}-\frac{1}{2}^*][q_1,p,q_2+1]}\,,
\end{aligned} \tag{A.6}$$

where by '$*$' we denote the $1/16$ semishortening. Consequently, we denote by $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}^*][q_1,p,q_2]}$ the $1/8$ semi-short multiplets appearing in the decomposition

$$\begin{aligned}
\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}][q_1,p,q_2]} \;=\; & \mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}^*][q_1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B-\frac{1}{2}}_{[j-\frac{1}{2}^*,\bar{\jmath}^*][q_1+1,p,q_2]} + \mathcal{V}^{\Delta+\frac{1}{2},B+\frac{1}{2}}_{[j^*,\bar{\jmath}-\frac{1}{2}^*][q_1,p,q_2+1]} + \\
& + \mathcal{V}^{\Delta+1,B}_{[j-\frac{1}{2}^*,\bar{\jmath}-\frac{1}{2}^*][q_1+1,p,q_2+1]}\,,
\end{aligned} \tag{A.7}$$

if left and right shortening conditions in (A.5) are simultaneously satisfied. The semishort multiplets appearing in this decomposition are constructed explicitly according to (A.2), (A.5).

Formulae (A.6) include the special cases $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][0,p,q_2]}$, $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][0,0,q_2]}$, and $\mathcal{V}^{\Delta,B}_{[j^*,\bar{\jmath}][0,0,0]}$, corresponding to (chiral) $1/8$, $3/16$, and $1/4$ semi-shortening, respectively; likewise for $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,p,0]}$, $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][q_1,0,0]}$, and $\mathcal{V}^{\Delta,B}_{[j,\bar{\jmath}^*][0,0,0]}$. For $j=0$ and $\bar{\jmath}=0$, respectively, the decompositions (A.6) yield negative spin labels. They are to be interpreted as BPS multiplets, denoted by '$\dagger$', as follows

$$\mathcal{V}^{\Delta,B}_{[-\frac{1}{2}^*,\bar{\jmath}][q_1,p,q_2]} \equiv \mathcal{V}^{\Delta+\frac{1}{2},B+\frac{1}{2}}_{[0^\dagger,\bar{\jmath}][q_1+1,p,q_2]}\,, \qquad \mathcal{V}^{\Delta,B}_{[j,-\frac{1}{2}^*][q_1,p,q_2]} \equiv \mathcal{V}^{\Delta+\frac{1}{2},B-\frac{1}{2}}_{[j,0^\dagger][q_1,p,q_2+1]}\,, \tag{A.8}$$

where one verifies that the BPS highest weight states satisfy the BPS shortening conditions of (A.5). In addition, there is the series $\mathcal{V}^{p,0}_{[0^\dagger,0^\dagger][0^\dagger,p,0^\dagger]}$ of $\frac{1}{2}$-BPS multiplets.

For convenience (but not quite accurately) we can also define

$$\mathcal{V}^{p,0}_{[-1^*,-1^*][0,p,0]} := \mathcal{V}^{p+2,0}_{[0^\dagger,0^\dagger][0^\dagger,p+2,0^\dagger]}\,. \tag{A.9}$$

B. Oscillator description

Here we collect some useful formulae, concerning the oscillator description of $\mathfrak{psu}(2,2|4)$ representations.

B.1 $\mathfrak{su}(2) \times \mathfrak{su}(2|4)$ invariant vacuum

The unphysical $\mathfrak{su}(2) \times \mathfrak{su}(2|4)$ invariant vacuum $|U\rangle$ is defined as the ground state of the set of bosonic $a_\alpha^{(s)}, b_{\dot\alpha}^{(s)}$ and fermionic oscillators $\theta_A^{(s)}$:

$$a_{\alpha,i}|U\rangle = b_{\dot\alpha}^{(s)}|U\rangle = \theta_A^{(s)}|U\rangle = 0\,, \tag{B.1}$$

with the vector index $s = 1,\ldots,L$ running over the sites in the SYM state and $\alpha, \dot\alpha = 1,2$ $A = 1,2,3,4$. Oscillators satisfy the usual creation-annihilation commutation relations:

$$\begin{aligned}\left[a_\alpha^{(s)}, a^\beta_{(s')}\right] &= \delta_{ss'}\delta_\alpha^\beta \qquad \left[b_{\dot\alpha}^{(s)}, b^{\dot\beta}_{(s)}\right] = \delta_{ss'}\delta_{\dot\alpha}^{\dot\beta}\,,\\ \left\{\theta_A^{(s)}, \theta^B_{(s')}\right\} &= \delta_{ss'}\delta_A^B\,.\end{aligned} \tag{B.2}$$

A SYM state with $\mathfrak{psu}(2,2|4)$ charges $[q_1,p,q_2][j,\bar\jmath]^{\Delta,B,L}$ can be constructed by acting on $|U\rangle$ with

$$\begin{aligned}\begin{pmatrix} n_{a_1}\\ n_{a_2}\end{pmatrix} &= \begin{pmatrix}\frac12\Delta+\frac12 B-\frac12 L+j\\ \frac12\Delta+\frac12 B-\frac12 L-j\end{pmatrix}, \quad \begin{pmatrix} n_{b_1}\\ n_{b_2}\end{pmatrix} = \begin{pmatrix}\frac12\Delta-\frac12 B-\frac12 L+\bar\jmath\\ \frac12\Delta-\frac12 B-\frac12 L-\bar\jmath\end{pmatrix},\\ \begin{pmatrix} n_{\theta_1}\\ n_{\theta_2}\\ n_{\theta_3}\\ n_{\theta_4}\end{pmatrix} &= \begin{pmatrix}\frac12 L-\frac12 B-\frac12 p-\frac34 q_1-\frac14 q_2\\ \frac12 L-\frac12 B-\frac12 p+\frac14 q_1-\frac14 q_2\\ \frac12 L-\frac12 B+\frac12 p+\frac14 q_1-\frac14 q_2\\ \frac12 L-\frac12 B+\frac12 p+\frac14 q_1+\frac34 q_2\end{pmatrix}.\end{aligned} \tag{B.3}$$

The $\mathfrak{psu}(2,2|4)$ charges can instead be read from the inverse relations:

$$\begin{aligned}\Delta &= L+\frac12 n_a+\frac12 n_b\,, \qquad B = L-\frac12 n_\theta = \frac12 n_a-\frac12 n_b\,,\\ [q_1,p,q_2] &= \left[n_{\theta_2}-n_{\theta_1}, n_{\theta_3}-n_{\theta_2}, n_{\theta_4}-n_{\theta_3}\right],\\ [j,\bar\jmath] &= \left[\frac12(n_{a_1}-n_{a_2}), \frac12(n_{b_1}-n_{b_2})\right].\end{aligned} \tag{B.4}$$

Physical states are defined by the vanishing central charge conditions:

$$n_a^{(s)} - n_b^{(s)} + n_\theta^{(s)} = 2\,. \tag{B.5}$$

at every site $s = 1,\ldots,L$.

B.2 Physical vacuum

The physical vacuum $|Z\rangle$ is defined as the ground state of the set of L species of bosonic $a_\alpha^{(s)}, b_{\dot\alpha}^{(s)}$ and fermionic oscillators $c_r^{(s)} = \theta_r^{(s)}$ and $d_{\dot p}^{(s)} = \bar\theta_{\dot p}^{(s)}$:

$$a_\alpha^{(s)}|Z\rangle = b_{\dot\alpha}^{(s)}|Z\rangle = c_p^{(s)}|Z\rangle = d_{\dot p}^{(s)}|Z\rangle = 0\,, \tag{B.6}$$

$\mathfrak{su}(4)$	$\mathfrak{su}(2)\times\mathfrak{su}(2)$
1	$\sum_{r=-2}^{2}[\ell-\frac{r}{2},\ell-\frac{r}{2}]+[\ell-1,\ell-1]+[\ell-1,\ell+1]$
4	$[\ell-\frac{1}{2},\ell-1]+[\ell-\frac{1}{2},\ell]+[\ell-1,\ell+\frac{1}{2}]+[\ell+1,\ell+\frac{1}{2}]$
6	$[\ell-1,\ell]+[\ell+\frac{1}{2},\ell-\frac{1}{2}]+[\ell,\ell+1]$
10	$[\ell+\frac{1}{2},\ell-\frac{1}{2}]$
15	$[\ell-\frac{1}{2},\ell-\frac{1}{2}]+[\ell,\ell+1]+[\ell,\ell]$
20	$[\ell-\frac{1}{2},\ell]$
20′	$[\ell,\ell]$

Table 4: Content of $\mathcal{V}^{2\ell,0}_{[000][\ell-1^*,\ell-1^*]}$ for $\ell\geq 2$.

with the vector index $s=1,\ldots,L$ running over the sites in the SYM state and $\alpha,\dot\alpha,p,\dot p=1,2$. Oscillators satisfy the usual creation-annihilation commutation relations:

$$\left[a^{(s)}_{\alpha},a^{\beta}_{(s')}\right]=\delta_{ss'}\delta^{\beta}_{\alpha}\,,\qquad \left[b^{(s)}_{\dot\alpha},b^{\dot\beta}_{(s')}\right]=\delta_{ss'}\delta^{\dot\beta}_{\dot\alpha}\,,$$
$$\left\{c^{(s)}_{p},c^{r}_{(s)}\right\}=\delta_{ss'}\delta^{r}_{p}\,,\qquad \left\{d^{(s)}_{\dot p},d^{\dot r}_{(s)}\right\}=\delta_{ss'}\delta^{\dot r}_{\dot p}\,. \tag{B.7}$$

A SYM state with $\mathfrak{psu}(2,2|4)$ charges $[q_1,p,q_2][j,\bar\jmath]^{\Delta,B,L}$ can be constructed by acting on $|ZZ\ldots Z\rangle$ with

$$\begin{pmatrix}n_{a_1}\\ n_{a_2}\end{pmatrix}=\begin{pmatrix}\frac12\Delta+\frac12B-\frac12L+j\\ \frac12\Delta+\frac12B-\frac12L-j\end{pmatrix},\qquad \begin{pmatrix}n_{b_1}\\ n_{b_2}\end{pmatrix}=\begin{pmatrix}\frac12\Delta-\frac12B-\frac12L+\bar\jmath\\ \frac12\Delta-\frac12B-\frac12L-\bar\jmath\end{pmatrix}, \tag{B.8}$$
$$\begin{pmatrix}n_{c_1}\\ n_{c_2}\end{pmatrix}=\begin{pmatrix}\frac12L-\frac12B-\frac12p-\frac34q_1-\frac14q_2\\ \frac12L-\frac12B-\frac12p+\frac14q_1-\frac14q_2\end{pmatrix},\qquad \begin{pmatrix}n_{d_1}\\ n_{d_2}\end{pmatrix}=\begin{pmatrix}\frac12L+\frac12B-\frac12p-\frac14q_1+\frac14q_2\\ \frac12L+\frac12B-\frac12p-\frac14q_1-\frac34q_2\end{pmatrix}.$$

$\mathfrak{psu}(2,2|4)$ charges can instead be read from the inverse relations:

$$\Delta=L+\frac12n_a+\frac12n_b\,,\qquad B=\frac12n_a-\frac12n_b\,,$$
$$[q_1,p,q_2]=[n_{c_2}-n_{c_1},L-n_{c_2}-n_{d_1},n_{d_1}-n_{d_2}]\,,$$
$$[j,\bar\jmath]=\left[\frac12(n_{a_1}-n_{a_2}),\frac12(n_{b_1}-n_{b_2})\right]. \tag{B.9}$$

Physical states are defined by the vanishing central charge conditions:

$$C^{(s)}=n^{(s)}_a-n^{(s)}_b+n^{(s)}_c-n^{(s)}_d=0 \tag{B.10}$$

at every site $s=1,\ldots,L$.

C. HS multiplets decomposition

For convenience of the reader we display here the translations of formulae (3.8).

$$\mathcal{Z}_{\square\square}=\sum_{n=0}^{\infty}\mathcal{V}^{2n,0}_{[-1+n^*,-1+n^*][0,0,0]}\,,$$

$$
\begin{aligned}
\mathcal{Z}_{\ydiagram{1,1}} &= \sum_{n=0}^{\infty} \mathcal{V}^{2n+1,0}_{[-\frac{1}{2}+n^*,-\frac{1}{2}+n^*][0,0,0]}\,, \\
\mathcal{Z}_{\ydiagram{3}} &= \sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{1+n,0}_{[-1+\frac{1}{2}n^*,-1+\frac{1}{2}n^*][0,1,0]} + \left(\mathcal{V}^{\frac{11}{2}+n,\frac{1}{2}}_{[\frac{3}{2}+\frac{1}{2}n^*,1+\frac{1}{2}n^*][0,0,1]} + \text{h.c.}\right)\right] + \\
&\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{4+4m+n,1}_{[1+2m+\frac{1}{2}n^*,\frac{1}{2}n][0,0,0]} + \mathcal{V}^{9+4m+n,1}_{[\frac{7}{2}+2m+\frac{1}{2}n^*,\frac{3}{2}+\frac{1}{2}n][0,0,0]} + \text{h.c.}\right]\,, \\
\mathcal{Z}_{\ydiagram{2,1}} &= \sum_{n=0}^{\infty} d_n \left[\mathcal{V}^{2+n,0}_{[\frac{1}{2}n-\frac{1}{2}^*,\frac{1}{2}n-\frac{1}{2}^*][0,1,0]} + \left(\mathcal{V}^{\frac{7}{2}+n,\frac{1}{2}}_{[\frac{1}{2}+\frac{1}{2}n^*,\frac{1}{2}n^*][0,0,1]} + \text{h.c}\right)\right] + \\
&\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} d_n \left[\mathcal{V}^{5+4m+n,1}_{[\frac{3}{2}+2m+\frac{1}{2}n^*,\frac{1}{2}+\frac{1}{2}n][0,0,0]} + \mathcal{V}^{7+4m+n,1}_{[\frac{5}{2}+2m+\frac{1}{2}n^*,\frac{1}{2}+\frac{1}{2}n][0,0,0]} + \text{h.c.}\right]\,, \\
\mathcal{Z}_{\ydiagram{1,1,1}} &= \sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{4+n,0}_{[\frac{1}{2}+\frac{1}{2}n^*,\frac{1}{2}+\frac{1}{2}n^*][0,1,0]} + \left(\mathcal{V}^{\frac{5}{2}+n,\frac{1}{2}}_{[\frac{1}{2}n^*,-\frac{1}{2}+\frac{1}{2}n^*][0,0,1]} + \text{h.c}\right)\right] + \\
&\quad + \sum_{m=0}^{\infty}\sum_{n=0}^{\infty} c_n \left[\mathcal{V}^{6+4m+n,1}_{[2+2m+\frac{1}{2}n^*,\frac{1}{2}n][0,0,0]} + \mathcal{V}^{7+4m+n,1}_{[\frac{5}{2}+2m+\frac{1}{2}n^*,\frac{3}{2}+\frac{1}{2}n][0,0,0]} + \text{h.c.}\right]\,. \qquad \text{(C.1)}
\end{aligned}
$$

In table 4 we rewrite the content of $\mathfrak{hs}(2,2|4)$ currents in the symmetric doubleton.

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